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Energy control of nonhomogeneous Toda lattices

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Abstract This paper presents a new approach to handle the problem of energy control of an n-degrees-of-freedom nonhomogeneous Toda lattice with fixed–fixed and fixed–free boundary conditions. The energy control problem is examined from an analytical dynamics perspective, and the theory of constrained motion is used to recast the energy control requirements on the Toda lattice as constraints on the mechanical system. No linearizations and/or approximations of the nonlinear dynamical system are made, and no a priori structure is imposed on the nature of the controller. Given the subset of masses at which control is to be applied, the fundamental equation of mechanics is employed to determine explicit closed-form expressions for the nonlinear control forces. The control provides global asymptotic convergence to any desired nonzero energy state provided that the first mass, or the last mass, or alternatively any two consecutive masses of the lattice are included in the subset of masses that are controlled. To illustrate the ease, simplicity, and efficacy with which the control methodology can be applied, numerical simulations involving a 101-mass Toda lattice are presented with control applied at various mass locations.

Keywords Fermi–Pasta–Ulam problem · Nonhomogeneous Toda chains · Energy control · Fundamental equation · Closed-form global asymptotic control · Actuator locations · Solitons

1 Introduction

Equipartition of energy in nonlinear lattices (or better known as the "FPU paradox") is a problem in nonlinear science that has mystified scientists since the seminal work done by Fermi, Pasta, and Ulam (FPU) in 1955 [1]. FPU considered a homogeneous, one-dimensional chain of masses wherein the nearest-neighboring masses were coupled using identical nonlinear spring elements. Instead of analyzing the detailed response of each of the masses in the nonlinear lattice, FPU concentrated their efforts on studying the total energy in the lattice. When excited in the lowest mode, the energy, instead of flowing from one mode of the system to another and eventually reaching a state of statistical equilibrium, kept periodically reverting back to the initial mode that they started it in, contrary to their expectations. Over fifty years of intensive research to

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resolve this paradox has led to numerous discoveries and opened many new avenues [2,3]. To date, the FPU problem is still an active area of research, and there are many questions related to its significance for science that still remain unanswered [4].

Motivated by the FPU paradox, in the 1960s, Toda [5] found analytical solutions to a homogeneous nonlinear lattice comprising of exponential spring elements, which is similar in nature to the FPU lattice. This nonlinear lattice is referred to as the "Toda Lattice" in the literature. Subsequently, in the 1970s, Ford [6], Henon [7], and Flaschka [8] independently established that the Toda lattice is an example of a completely integrable Hamiltonian system. But perhaps the most interesting aspect of Toda lattices is the fact that they admit multiple soliton solutions [5]. Besides their obvious importance in theoretical physics, Toda lattices also find many practical engineering applications primarily due to the disparate nature of the tensile and compressive forces of its elastic spring elements, which arises from an asymmetry in its potential. In nature, it is often difficult to find a mechanical system that has perfectly linear spring elements. One often finds that the spring elements are stronger in tension and weaker under compression, or vice versa. Many elastic materials also exhibit this asymmetrical behavior under tensile and compressive loading. For example, flexible cables in suspension bridges are stronger under tensile forces and weaker under compressive forces, and Toda lattices can potentially be used in the modeling of such systems.

The main focus of the present study is to control the energy of a *nonhomogeneous* Toda lattice and bring it to a desired energy level. Control of Toda lattices has received sparse attention in the literature. Puta and Tudoran address the problem of controllability of a homogeneous, n-degrees-of-freedom Toda lattice and find that the lattice is controllable since it satisfies the Lie algebra rank condition [9]. Schmidt et al. studied the observability properties of an n-periodic homogeneous Toda lattice and showed that the lattice is globally observable for any number of masses in the lattice by using a single momentum and the exponential of the distance between that mass and a neighboring mass as output [10]. Palamakumbura et al. consider an n-periodic homogeneous Toda lattice in terms of Flaschka variables and use full-state feedback as well as local feedback control to drive the solution of the controlled system to any solution of the uncontrolled system such as solitons and phonons [11].

Nearly all the research done to date in this field has been focused on lattices with identical masses and identical spring elements (i.e., homogeneous lattices). Work on nonhomogeneous lattices is indeed scant and seems limited to two-degrees-of-freedom systems [12–14]. Analytical results on both the dynamics and the control of large-amplitude nonlinear motion of nonhomogeneous lattices are, to the best of the authors' knowledge, nonexistent. Nevertheless, nonhomogeneous lattices are more representative of real-life behavior when compared to their idealized homogeneous counterparts. And, therefore, in this study, we consider an n-degrees-of-freedom nonhomogeneous Toda lattice with fixed–fixed and fixed–free boundary conditions. The problem of energy control of Toda lattices with fixed–fixed ends has been previously attempted by Polushin [15], but again this work deals with homogeneous lattices. Besides dealing with a nonhomogeneous Toda lattice, the control approach developed here is widely different from that used in [11] and [15].

More specifically, this paper distinguishes itself from previous work on the energy control of Toda lattices in four key ways—firstly, the Toda lattice under consideration is nonhomogeneous (i.e., the lattice is made up of dissimilar masses and dissimilar spring elements along its length) as compared to the homogeneous lattice considered in nearly all the current literature. Both fixed–fixed and fixed–free boundary conditions are studied. Secondly, the problem of energy control of Toda lattices is approached from an analytical dynamics perspective, and the theory of constrained motion is used to recast the control requirements as constraints on the dynamical system. The fundamental equation of mechanics [16] is employed to obtain a closed-form expression of the explicit nonlinear control force. Thirdly, the methodology developed herein allows us to explicitly determine the control forces needed to be applied to any arbitrarily chosen subset of masses that are designated to have control inputs and still achieve global asymptotic convergence to any desired nonzero energy state provided that the first mass, or the last mass, or alternatively any two consecutive masses of the nonhomogeneous lattice are included in this subset. Lastly, once the energy of the system is brought to its desired value, the control forces automatically terminate, and the conservative nature of the ensuing Hamiltonian dynamics is utilized to main-

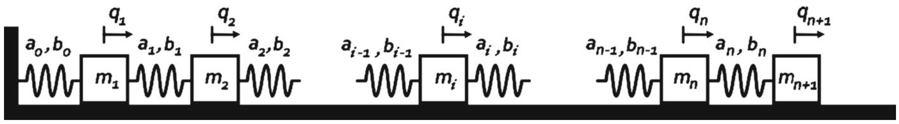

Fig. 1 Finite degrees-of-freedom nonhomogeneous Toda lattice

tain the system's energy at the desired level for all future time.

It is interesting to note that although it is extremely difficult to understand the dynamical response of the nonhomogeneous Toda lattice and obtain anything nearing general closed-form analytical solutions for the response—a consequence of which is the virtually nonexistent literature on the subject—its energy control, however, can be done relatively easily, and the necessary control forces can be obtained in closed form. This seems to beg the question: Has Nature somehow intentionally made it easier for us to control the energy of nonlinear systems rather than to determine their exact nonlinear behavior?

The paper is organized as follows. In Sect. 2, an introduction to the physics of the n-degrees-of-freedom nonhomogeneous Toda lattice is presented. The constrained motion approach is briefly recalled in Sect. 3. In Sect. 4, the energy control problem in Toda lattices is formulated, and a closed-form expression for the nonlinear control force is derived. In Sect. 5, the invariance principle [17] is used to derive sufficient conditions for the placement of the actuators, so that the control force obtained in Sect. 4 gives us global asymptotic convergence to any desired nonzero energy state. Finally, in Sect. 6, numerical simulations involving a 101-mass Toda lattice with fixed–fixed and fixed–free boundary conditions are presented that illustrate the ease and efficacy with which the control methodology can be applied. Several of the technical details have been placed in the Appendices in order to maintain the flow of thought.

2 Physics of the Toda lattice

The Toda lattice is a simple model for a nonlinear one-dimensional crystal that describes the motion of a chain of particles with exponential interactions between the nearest-neighboring elements (see Fig. 1) [5].

Consider a single-degree-of-freedom (SDOF) spring-mass system with Toda spring stiffness that is

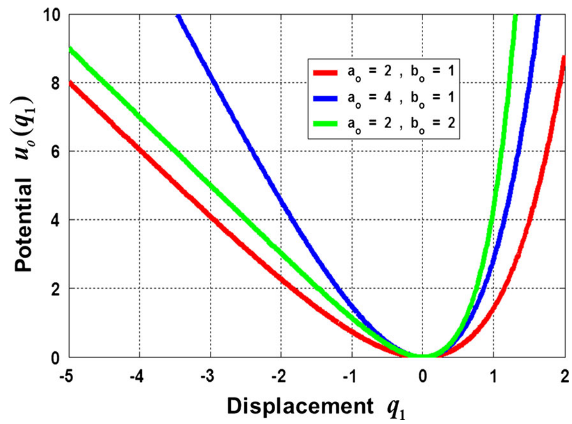

Fig. 2 Exponential potential $a_o,\ b_o > 0$. (Color figure online)

obtained by setting $m_i = 0\ \forall\ i = 2, 3, \ldots, n+1$ (see Fig. 1). The expression for the nonlinear potential (Fig. 2) of the Toda spring is given by the smooth and continuously differentiable function

$$u_o(q_1) = \frac{a_o}{b_o} e^{b_o q_1} - a_o q_1 - \frac{a_o}{b_o}, \quad a_o > 0,\ b_o > 0, \tag{2.1}$$

where the displacement, q_1, of the mass, m_1, is measured from its equilibrium position in an inertial frame of reference. A constant $(-a_o/b_o)$ has been included in the expression of the potential to ensure that $u_o(0) = 0$. The potential has a single stationary point at $q_1 = 0$ and since $u_o''(0) = a_o b_o > 0$, $q_1 = 0$ is a global minimum of (2.1). Further, since $u_o' = a_o(e^{b_o q_1} - 1) > 0\ \forall\ q_1 > 0$ and $u_o' < 0\ \forall\ q_1 < 0$, u_o is strictly increasing in the interval $0 < q_1 < \infty$ and is strictly decreasing in the interval $-\infty < q_1 < 0$. Consequently, the potential function is strictly radially increasing (see Fig. 2) and hence radially unbounded [18]. It is also strictly positive definite with $u_o(0) = 0$ and $u_o(q_1) > 0\ \forall\ q_1 \neq 0$. Further, since $u_o''(q_1) = a_o b_o e^{b_o q_1} > 0\ \forall\ q_1$, the potential of the Toda spring (2.1) is a strictly convex function. The exponential Toda spring force $F_s(q)$ for this SDOF spring-mass system is given by

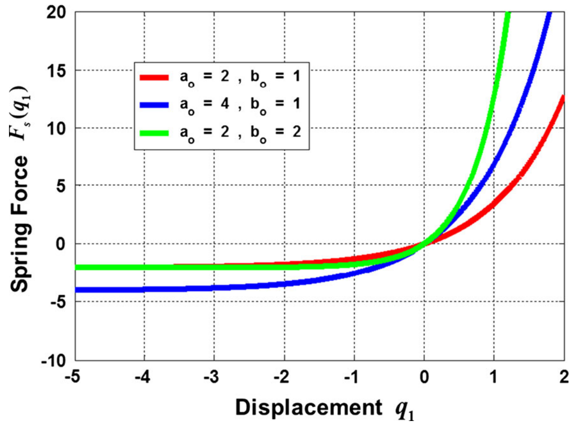

Fig. 3 Spring force of the Toda lattice. (Color figure online)

$$F_s(q_1) = -F_{\text{restoring}}(q_1) = \frac{\partial u_o(q_1)}{\partial q_1} = a_o\left(e^{b_o q_1} - 1\right) \tag{2.2}$$

For sufficiently small q_1, the spring force is approximately linear. However, the nonlinearity of the force gains prominence as q_1 increases. Also, a larger force is required to stretch the spring by a unit distance than is required to compress it (see Fig. 3). Hence, the Toda lattice considered in this paper possesses spring elements that are stronger in tension than in compression. Such systems arise frequently in structural subsystems such as the stringers in suspension bridges. In the present study, although we focus our attention on a Toda spring that is strong in tension and weak under compression ($a > 0,\ b > 0$), the theory developed herein is also, in general, applicable to a Toda spring that is weak in tension and strong under compression ($a < 0,\ b < 0$).

2.1 Equations of motion of the Toda lattice

Consider an n-degrees-of-freedom (NDOF) nonhomogeneous undamped Toda lattice (as shown in Fig. 1), in which the mass at the ith location is denoted by m_i. The kinetic energy of this lattice can be written as

$$T(\dot{q}) = \sum_{i=1}^{n+1} \frac{1}{2} m_i \dot{q}_i^2, \tag{2.3}$$

where $\dot{q}_i$ denotes the velocity of the ith mass in the lattice. The potential energy of the lattice is composed of exponential interactions between the nearest-neighboring elements and is defined by

$$\begin{aligned} U(q) &= \sum_{i=0}^{n} u_i(q_{i+1} - q_i) \\ &= \sum_{i=0}^{n} \left[\frac{a_i}{b_i} e^{b_i(q_{i+1}-q_i)} - a_i\,(q_{i+1} - q_i) - \frac{a_i}{b_i}\right], \end{aligned} \tag{2.4}$$

where $a_i,\ b_i > 0$ denote the spring constants of the ith spring element in the lattice and $q_o \equiv 0$ because the left end of the lattice is always fixed (see Fig. 1). The total energy H of the Toda lattice is a smooth and continuously differentiable function and is given by

$$\begin{aligned} H(q, \dot{q}) = T(\dot{q}) + U(q) = \sum_{i=1}^{n+1} \left[\frac{1}{2} m_i\, \dot{q}_i^2\right] \\ + \sum_{i=0}^{n} \left[\frac{a_i}{b_i} e^{b_i(q_{i+1}-q_i)} - a_i\,(q_{i+1} - q_i) - \frac{a_i}{b_i}\right], \end{aligned} \tag{2.5}$$

The energy H is a positive definite function with $H(0, 0) = 0$, and $H(q, \dot{q}) > 0$ for all $q, \dot{q} \neq 0$ (see Appendix 1). The equations of motion of the nonhomogeneous n-degrees-of-freedom (NDOF) Toda lattice can be derived using Newton's laws of motion and are given by

$$m_i\, \ddot{q}_i = a_i\left[e^{b_i(q_{i+1}-q_i)} - 1\right] - a_{i-1}\left[e^{b_{i-1}(q_i - q_{i-1})} - 1\right]; \quad i = 1, 2, \ldots, n+1, \tag{2.6}$$

where $q_{n+1} \equiv 0$ for an n-mass lattice with fixed–fixed boundary conditions and $a_n = b_n = 0$ for an n-mass lattice with fixed–free boundary conditions as there is no spring between the masses m_n and m_{n+1} (see Fig. 1). When Eq. (2.6) is expressed in matrix form, the equation of motion of a nonhomogeneous NDOF Toda lattice with fixed–fixed boundary conditions is given by $M\,\ddot{q} = F(q)$, which when written explicitly yields

$$\begin{bmatrix} m_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & m_i & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & m_n \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ \ddot{q}_i \\ \vdots \\ \ddot{q}_n \end{bmatrix}$$

$$= \begin{bmatrix} a_1\left[e^{b_1(q_2-q_1)}-1\right]-a_0\left[e^{b_0(q_1)}-1\right] \\ \vdots \\ a_i\left[e^{b_i(q_{i+1}-q_i)}-1\right]-a_{i-1}\left[e^{b_{i-1}(q_i-q_{i-1})}-1\right] \\ \vdots \\ a_n\left[e^{b_n(-q_n)}-1\right]-a_{n-1}\left[e^{b_{n-1}(q_n-q_{n-1})}-1\right] \end{bmatrix}, \tag{2.7}$$

where the n-by-n mass matrix M is diagonal, and the column vector of generalized "given" forces F is given by the right-hand side of Eq. (2.7). Furthermore, by setting $a_n = b_n = 0$ in Eq. (2.7), we obtain the matrix representation for the equation of motion of a nonhomogeneous NDOF Toda lattice with fixed–free boundary conditions.

While the analysis here is restricted to spring forces whose potentials are given by Eq. (2.1), what follows would also be applicable to systems in which the spring forces have polynomial nonlinearities (as often found in engineering systems) provided their potentials satisfy certain properties. Furthermore, we assume that there is no damping since the inclusion of damping adds complexities that go beyond the current scope of this paper.

3 Constrained motion approach and the fundamental equation of mechanics

In this paper, we use the fundamental equation of mechanics [19–26] to derive the constrained (controlled) equations of motion of the Toda lattice and thus to obtain the explicit nonlinear control forces that are required to achieve the desired energy stabilization. The fundamental equation is known for the relative ease with which the constrained equations of motion of a complex multibody system can be derived in comparison with other classical methods. A report on the numerical efficiency of this formulation in multibody dynamics has been presented by de Falco [23].

Consider an unconstrained [19], discrete dynamic system of n particles (similar to our NDOF Toda lattice with appropriate boundary conditions as described in Sect. 2). The equations of motion of this unconstrained system at a certain instant of time t can be written down using Newton's laws or Lagrange's method as

$$M(q,t)\ddot{q} = F(q,\dot{q},t), \quad q(0) = q_o,\ \dot{q}(0) = \dot{q}_o, \tag{3.1}$$

where M is the n-by-n symmetric, positive definite mass matrix, q is the n-vector of generalized coordinates of the system, and F is the n-vector of generalized "given" forces acting on the unconstrained system. The acceleration a of the unconstrained system (3.1) is given by

$$a(q,\dot{q},t) = [M(q,t)]^{-1} F(q,\dot{q},t). \tag{3.2}$$

Consider now that we impose a set of m constraints on the unconstrained system (3.1), all of which may or may not be independent (i.e., some of the constraints may be the combination of others) [25].

$$\phi_i(q,\dot{q},t) = 0, \quad i = 1, 2, 3 \ldots m. \tag{3.3}$$

The initial conditions stated in (3.1) are assumed to satisfy these constraint equations. However, in some cases, it may not be possible to initialize the unconstrained system from points in the phase space where the constraints are satisfied. Thus, instead of considering the existing set of m constraints described by Eq. (3.3), we modify these constraint equations as follows [20].

$$\Psi_i(q,\dot{q},\ddot{q},t) = \dot{\phi}_i + \beta\phi_i = 0, \quad i = 1, 2, 3 \ldots m, \tag{3.4}$$

where $\beta(q,\dot{q}) > 0$ is chosen such that the system of equations (3.4) has an equilibrium point described by (3.3), and that this equilibrium point is stable. This set of m modified constraints can now be expressed in the general constraint matrix form as

$$A(q,\dot{q},t)\ddot{q} = b(q,\dot{q},t), \tag{3.5}$$

where A is an m-by-n constraint matrix of rank r (i.e., r out of the m constraint equations are independent), while b is a column vector with m entries. The presence of constraints causes the acceleration of the constrained system to deviate from its unconstrained acceleration at every instant of time t. This deviation in the acceleration of the constrained system is brought about by a force F^C, called the constraint force, which is exerted on the system by virtue of the fact that the unconstrained system must now further satisfy an additional set of constraints. The explicit equations of motion of the constrained system can now be written down as

$$M(q,t)\ddot{q} = F(q,\dot{q},t) + F^C(q,\dot{q},t), \tag{3.6}$$

where F^C is the set of additional forces that arise by virtue of the application of the m constraints. One can also envision F^C to be the set of control forces that are required to be applied to the uncontrolled openloop system (unconstrained system) to obtain the controlled closed-loop system (constrained system) [21]. Udwadia and Kalaba [19,22,26] proposed the following closed-form expression for the control (constraint) force

$$F^C(q, \dot{q}, t) = M^{1/2}(AM^{-1/2})^+(b - Aa), \qquad (3.7)$$

where $(AM^{-1/2})^+$ denotes the Moore–Penrose inverse of the matrix $(AM^{-1/2})$. Equation (3.6) along with (3.7) is referred to as the "fundamental equation of mechanics." Equation (3.7) provides the optimal set of control forces that minimize the control cost given by $J(t) = [F^C]^T M^{-1}[F^C]$ at each instant of time while causing the constraints to be exactly satisfied [24]. Once the matrices *M, F, A, b* are obtained from the description of the unconstrained system and the constraint equations, the constraint (control) force can be readily calculated using Eq. (3.7). This often greatly reduces the conceptualization effort when compared with other classical methods.

The generality of this formulation makes it applicable in many diverse areas of mechanics. For example, see Udwadia and Han [27], and Mylapilli [28] for an application of this formulation to the problem of motion synchronization of multiple coupled (or uncoupled) chaotic gyroscopes. Further applications of this formulation to rotational dynamics, satellite control, and uncertain mechanical systems can be found in Refs. [29,30] and [31], respectively.

The papers in Refs. [27–31] primarily deal with the use of the fundamental equation of mechanics applied to the full-state control of nonlinear, nonautonomous systems that have a relatively small number of degrees of freedom. This paper deals with the use of Lyapunov stability theory in conjunction with the fundamental equation of mechanics to investigate highly *underactuated*, global asymptotic energy control of autonomous systems that can have a very large number of degrees of freedom (see illustrative examples in Sect. 6).

4 Solution to the energy control problem

Consider a nonhomogeneous, n-mass Toda lattice with energy $H > 0$ and appropriate boundary conditions (see Eqs. 2.6 and 2.7), which we henceforth refer to as the unconstrained Toda lattice. The energy control problem for this unconstrained system is formulated as follows.

Given a set of k masses selected from among the n masses of the lattice, find the explicit control forces that need to be applied to this set of k masses such that the total energy of the Toda lattice approaches a given positive value H^* as $t \to +\infty$

$$\text{i.e., } H(q(t), \dot{q}(t)) \to H^* \quad \text{as} \quad t \to +\infty, \quad H^* > 0 \qquad (4.1)$$

Although we assume at this stage that the locations of these k actuators (where $1 \le k \le n$) can be arbitrarily selected from among the n masses in the lattice, we will later show that in order to have global asymptotic convergence to any given nonzero desired energy state H^*, the set of actuator locations need to satisfy certain conditions when $k < n$ and the system is underactuated (see Sect. 5.2).

In this paper, the energy control problem (4.1) is approached from a constrained motion perspective, which comprises of three vital steps [19]. The first step involves the derivation of the equations of motion of the unconstrained system. For a nonhomogeneous NDOF Toda lattice, this has already been discussed in Sect. 2. The second step involves the formulation of the constraint equations (see Sect. 4.1), and the last step deals with the use of the fundamental equation (Eqs. 3.6 and 3.7) to obtain the constrained equations of motion of the Toda lattice (see Sect. 4.2). In the process, we also find a closed-form expression for the explicit nonlinear control force that is required to be applied to the uncontrolled system (2.6) to stabilize the energy of the Toda lattice at the desired level.

4.1 Formulation of the energy constraints

Consider an unconstrained, nonhomogeneous NDOF Toda lattice with appropriate boundary conditions. The unconstrained acceleration $a(q)$ of the Toda lattice with fixed–fixed ends can be computed as

$$a(q) = M^{-1}F(q)$$
$$= \begin{bmatrix} \frac{a_1}{m_1}\left[e^{b_1(q_2-q_1)}-1\right] - \frac{a_o}{m_1}\left[e^{b_o(q_1)}-1\right] \\ \vdots \\ \frac{a_i}{m_i}\left[e^{b_i(q_{i+1}-q_i)}-1\right] - \frac{a_{i-1}}{m_i}\left[e^{b_{i-1}(q_i-q_{i-1})}-1\right] \\ \vdots \\ \frac{a_n}{m_n}\left[e^{b_n(-q_n)}-1\right] - \frac{a_{n-1}}{m_n}\left[e^{b_{n-1}(q_n-q_{n-1})}-1\right] \end{bmatrix}. \qquad (4.2)$$

The unconstrained acceleration of the fixed–free Toda lattice can be similarly obtained by setting $a_n = b_n = 0$ in Eq. (4.2). Suppose now that out of these n masses, we apply control inputs to k arbitrarily selected masses, where $1 \le k \le n$. The locations of these k masses where a control input is applied are denoted by the ordered set

$$S_C = \{i_1, i_2, i_3, \ldots i_k\}, \tag{4.3}$$

where, with no loss of generality, we order these locations along the Toda lattice, so that $i_1 < i_2 < i_3 < \ldots < i_k$. Similarly, the set of $(n-k)$ masses at which no control is applied is given by the complement of the set S_C which we denote by

$$\begin{aligned} S_N = S_C^c &= \{1, 2, 3 \ldots n\} \backslash \{i_1, i_2, i_3, \ldots i_k\} \\ &= \{j_1, j_2, j_3, \ldots j_{n-k}\}, \end{aligned} \tag{4.4}$$

where again $j_1 < j_2 < j_3 < \ldots < j_{n-k}$. It is also convenient to represent this information in terms of matrices. The following matrices are defined to simplify the notation.

(1) The mass matrices associated with the set of controlled and uncontrolled masses are represented by $M_C = \operatorname{diag}\left(m_{i_1}, m_{i_2}, m_{i_3}, \ldots m_{i_k}\right)$ and $M_N = \operatorname{diag}\left(m_{j_1}, m_{j_2}, m_{j_3}, \ldots m_{j_{n-k}}\right)$, respectively.

(2) The displacements associated with the set of controlled and uncontrolled masses are represented by the column vectors $q_C = \left[q_{i_1}\ q_{i_2}\ \ldots q_{i_k}\right]^T$ and $q_N = \left[q_{j_1}\ q_{j_2}\ \ldots q_{j_{n-k}}\right]^T$, respectively.

(3) A k-by-n "control selection matrix," C, is defined such that every element of its gth row $(1 \le g \le k)$ is zero except for the i_gth element (where $i_g \in S_C$) which is unity.

(4) Similarly, we define an $(n-k)$-by-n "no-control selection matrix," N, such that every element of its hth row $(1 \le h \le n-k)$ is zero except for the j_hth element (where $j_h \in S_N$) which is unity.

(5) We note that the n-by-n diagonal matrix $I_C = C^T C$ has zeroes all along its diagonal except for the (i_g, i_g) elements which are unity $(i_g \in S_C)$.

(6) Similarly, the n-by-n diagonal matrix $I_N = N^T N$ has zeroes all along its diagonal except for the (j_h, j_h) elements which are unity $(j_h \in S_N)$.

While dealing with the energy control problem, we interpret the energy requirements as an energy constraint on the unconstrained Toda lattice (2.6).

1. *Constraint of "energy stabilization"* Using Eq. (2.5), the energy stabilization constraint is given by

$$\begin{aligned} \phi(q, \dot{q}) &= H(q, \dot{q}) - H^* \\ &= \left(\frac{1}{2}\dot{q}^T M \dot{q} + U(q)\right) - H^* = 0, \end{aligned} \tag{4.5}$$

where $H(q, \dot{q})$ is rewritten in matrix-vector notation, and H^* denotes the given nonzero desired energy state of the system. Equation (4.5) resembles constraint equation (3.3) and therefore needs to be differentiated once with respect to time, so that it can be expressed in the general form of Eq. (3.5). Further, a modified constraint equation [20] is generated by introducing $\beta > 0$ (see Eq. 3.4), so that the Toda lattice can be initiated from any arbitrary initial energy state. The modified energy stabilization constraint can now be expressed as

$$\begin{aligned} \Psi(q, \dot{q}, \ddot{q}) &= \frac{\mathrm{d}}{\mathrm{d}t}(\phi) + \beta\phi = 0 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{1}{2}\dot{q}^T M \dot{q} + U(q)\right) - H^*\right] \\ &\quad + \beta\left(H - H^*\right) = 0 \\ &= \frac{1}{2}\dot{q}^T\left(M + M^T\right)\frac{\mathrm{d}\dot{q}}{\mathrm{d}t} + \left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^T\left(\frac{\partial U}{\partial q}\right) \\ &\quad + \beta\left(H - H^*\right) = 0 \\ &= \dot{q}^T M \ddot{q} - \dot{q}^T F + \beta\left(H - H^*\right) = 0 \end{aligned} \tag{4.6}$$

2. *Constraint of "no-control"* In addition to the energy stabilization constraint, a constraint of "no-control" is imposed on all the masses that belong to the set S_N that are left unactuated. Since no control is being applied to these masses, the prevailing unconstrained motion of these masses (2.6) can themselves be considered as constraints. Thus, this set of $(n-k)$ "no-control" constraints can be described in matrix form as

$$N(M\ddot{q} - F) = 0. \tag{4.7}$$

When the constraints described by Eqs. (4.6) and (4.7) are expressed in the general constraint matrix form (see Eq. 3.5), this leads to an $(n-k+1)$-by-n constraint matrix A given by

$$A = \begin{bmatrix} \dot{q}^T M \\ N M \end{bmatrix} = \begin{bmatrix} \dot{q}^T \\ N \end{bmatrix} M, \tag{4.8}$$

and an $(n - k + 1)$-sized column vector b given by

$$b = \begin{bmatrix} \dot{q}^T F - \beta (H - H^*) \\ N F \end{bmatrix}. \tag{4.9}$$

4.2 Equations of motion of the constrained Toda lattice system

Once the matrices M, F, a, A, and b are known for an NDOF Toda lattice, the explicit nonlinear control force F^C can be computed using Eq. (3.7). A detailed derivation of the control force can be found in Appendix 2. The control force in closed form is given by

$$F^C(q, \dot{q}) = \frac{-\beta (H(q, \dot{q}) - H^*)}{\dot{q}_C^T M_C \dot{q}_C} I_C M \dot{q}, \tag{4.10}$$

where $\dot{q}_C^T M_C \dot{q}_C = \sum_{g=1}^{k} \left(m_{i_g} \dot{q}_{i_g}^2 \right)$. The control force (4.10) possesses a singularity when the velocities of the set of masses that are controlled are all simultaneously zero. To avoid this, we choose β as

$$\beta (q, \dot{q}) = \left(\dot{q}_C^T M_C \dot{q}_C \right) \cdot \lambda (q, \dot{q}), \tag{4.11}$$

where $\lambda (q, \dot{q}) > 0$. Moreover, for simplicity, we choose $\lambda(q, \dot{q}) = \lambda_o$, where λ_o is a positive constant that can be suitably altered to control the rate at which the system converges to the desired energy state H^*. With this simplification, the explicit control force is now given by

$$\begin{aligned} F^C &= -\lambda_o \left(H(q, \dot{q}) - H^* \right) I_C M \dot{q} \\ &= -f(q, \dot{q}) I_C M \dot{q}. \end{aligned} \tag{4.12}$$

Though it might appear that the control force, which depends linearly on the momentum of the controlled masses, resembles a velocity feedback type of control, the nonlinear gain $f(q, \dot{q})$ qualitatively changes the nature of the feedback, as shown later on. The equations of motion of the controlled (constrained) Toda lattice with appropriate boundary conditions can now be written using Eq. (3.6), where the "given" force F is obtained from the uncontrolled system (see Eqs. 2.6 and 2.7), and the control force (constraint force) F^C is explicitly given by Eq. (4.12).

5 Global asymptotic convergence to the energy state H^* in $\mathbb{R}^{2n} - \{O\}$

In this section, our aim is to prove that the control force F^C gives us global asymptotic convergence to any given desired energy state H^* in $\mathbb{R}^{2n} - \{O\}$ provided that the first mass, or the last mass, or alternatively any two consecutive masses of the Toda lattice are included in the subset of masses that are controlled. The energy of the uncontrolled system is assumed to be greater than zero. To acquire insight into the nature of the control, we first consider a single-degree-of freedom (SDOF) spring-mass oscillator with Toda spring stiffness and later generalize these results to a nonhomogeneous NDOF Toda lattice with fixed–fixed (and fixed–free) boundary conditions. Proofs related to the NDOF system are derived in detail in the Appendices.

5.1 SDOF spring-mass system with Toda stiffness

Consider a SDOF spring-mass system with Toda spring stiffness as discussed in Sect. 2. The uncontrolled (unconstrained) equation of motion of this spring-mass system with unit mass is given by

$$\begin{aligned} &\ddot{q}_1 + a_o(e^{b_o q_1} - 1) = 0, \\ &q_1(0) = q_1^{(0)}, \ \dot{q}_1(0) = \dot{q}_1^{(0)}, \end{aligned} \tag{5.1}$$

where the initial displacement and the initial velocity of the system are specified by $q_1^{(0)}$ and $\dot{q}_1^{(0)}$, respectively. Let us represent this SDOF system (5.1) in an equivalent state-space form:

$$\begin{aligned} &\dot{X}_1 = X_2, \\ \text{and} \quad &\dot{X}_2 = -a_o(e^{b_o X_1} - 1), \end{aligned} \tag{5.2}$$

where $X_1 = q_1$ and $X_2 = \dot{q}_1$. The SDOF system (5.2) has a single isolated equilibrium point at the origin O, which is a center [32]. Hence, the phase space of uncontrolled system (5.2) is composed of concentric closed orbits around the origin with each closed orbit denoting a constant energy level as shown in Fig. 4a.

Consider now that this uncontrolled system (5.1) is subjected to an energy stabilization constraint (4.6). The control force can be computed using Eq. (4.12) and is given by

$$F^C = -\lambda_o \left(H(q_1, \dot{q}_1) - H^* \right) \dot{q}_1, \quad \lambda_o > 0 \tag{5.3}$$

where,

$$H (q_1, \dot{q}_1) = \frac{1}{2}\dot{q}_1^2 + \frac{a_o}{b_o} e^{b_o q_1} - a_o q_1 - \frac{a_o}{b_o}. \tag{5.4}$$

The energy H of the SDOF Toda oscillator is positive definite (see Appendix 1) and radially unbounded (Appendix 4). Also, H increases monotonically in

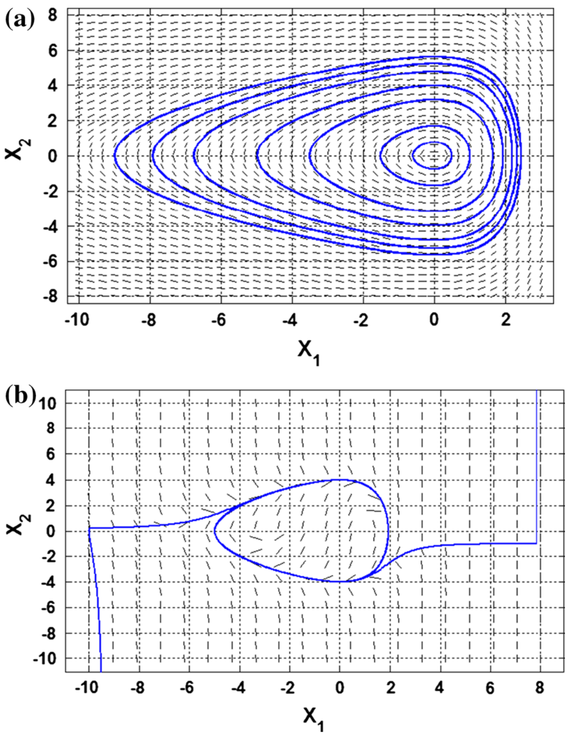

Fig. 4 **a** 2D phase portrait of the uncontrolled SDOF oscillator with Toda spring stiffness ($a_o = 2,\ b_o = 1$). **b** 2D phase portrait of the controlled SDOF oscillator with Toda spring stiffness ($a_o = 2,\ b_o = 1,\ \lambda_o = 1,\ H^* = 8$). Two different trajectories starting from two different initial conditions are shown. (Color figure online)

every radial direction from the origin. Therefore, any constant energy curve is a closed orbit in phase space. With the control force (5.3) at our disposal, the controlled (constrained) equations of motion of the SDOF Toda oscillator can now be written as

$$\ddot{q}_1 + \lambda_o \left(H(q_1, \dot{q}_1) - H^*\right) \dot{q}_1 + a_o(e^{b_o q_1} - 1) = 0. \tag{5.5}$$

Eq. (5.5) resembles the familiar form of a self-excited nonlinear oscillator with a nonlinear damping term similar to those found in Van der Pol-type systems. When $H > H^*$, the damping in the system is positive, and the energy of the system is lowered. Conversely, when $H < H^*$, the damping is negative, and the energy of the system is raised. When $H = H^*$ is attained, the control force terminates, and the conservative nature of the lattice is utilized to maintain its energy at H^* for all future time.

The equivalent pair of first-order equations for the constrained system (5.5) is given by

$$\dot{X}_1 = X_2,$$

$$\dot{X}_2 = -a_o(e^{b_o X_1} - 1) - \lambda_o \left(H(X_1, X_2) - H^*\right) X_2. \tag{5.6}$$

The constrained system (5.6) still possesses a single isolated equilibrium point at the origin, but it is now unstable. The introduction of the control force (5.3) has destroyed the concentric closed orbits of the uncontrolled (unconstrained) system (see Fig. 4a) and has led to the creation of an unstable origin as well as a stable limit cycle (described by $H(X_1, X_2) = H^*$) in the controlled (constrained) system (see Fig. 4b). As shown in this figure, the controlled system asymptotically tends to the manifold $H = H^*$ in the two-dimensional phase space, and to get to this desired manifold, it can take one of many different trajectories depending on the initial conditions of the controlled system and the value chosen for the parameter $\lambda_o > 0$.

We now investigate whether we can analytically establish the convergence of all orbits in $\mathbb{R}^2 - \{O\}$ to the manifold $H(X_1, X_2) = H^*$ for the controlled system. And to do this, we resort to Lasalle's invariance principle [17].

Invariance principle Lasalle's invariance principle [17] in $\mathbb{R}^n$ is postulated as follows.

Let Ω be a compact set (Ω is a subset of D where $D \subset \mathbb{R}^n$) that is positively invariant. Let $V : D \to \mathbb{R}$ be a continuously differentiable scalar function such that $\dot{V} \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V} = 0$. Let P be the largest invariant set in E. Then, every solution $x(t)$ starting in Ω approaches P as $t \to \infty$.

Consider a continuously differentiable scalar function V given by

$$V(X_1, X_2) = \frac{1}{2}\left(H(X_1, X_2) - H^*\right)^2, \quad H^* > 0, \tag{5.7}$$

defined on the set Ω described by

$$\Omega = \left\{(X_1, X_2) \in \mathbb{R}^2 \,|\, \varepsilon \leq H\,(X_1, X_2) \leq c\right\}, \tag{5.8}$$

where $0 < \varepsilon < H^* < c$, and H is the energy of the system described by (5.4). By choosing $\varepsilon > 0$, an open region around the origin O ($X_1 = X_2 = 0$) is excluded from the set Ω. Our basic motive in choosing Ω as in (5.8) is to establish that the origin O is an unstable fixed point, and all trajectories in $\mathbb{R}^2 - \{O\}$ asymptotically converge to the closed periodic orbit $H(X_1, X_2) = H^*$.

1. *Ω is compact set* A formal proof of this result is presented in Appendix 4. However, given that constant

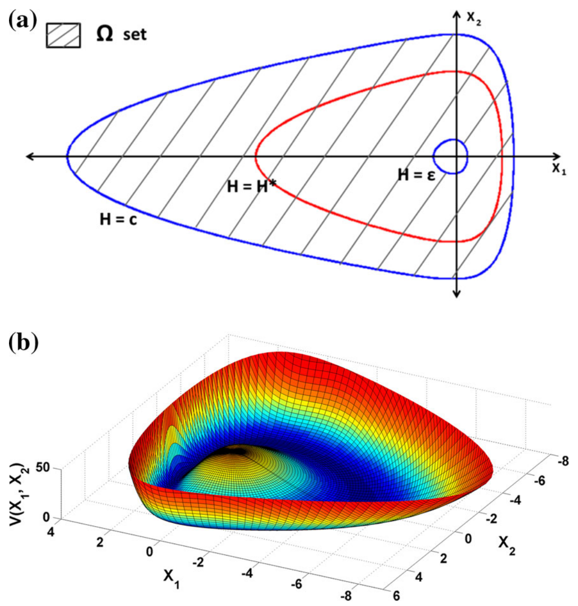

Fig. 5 **a** Pictorial representation of a typical Ω set. **b** scalar energy error V plotted as a function of phase variables X_1 and X_2 ($a_o = 2,\ b_o = 1,\ H^* = 8$). (Color figure online)

energy curves of (5.1) form closed orbits in $\mathbb{R}^2$, from Fig. 5a it is easy to see that the set Ω is indeed closed and bounded and therefore compact.

2. Ω *is positively invariant* A set W is said to be positively invariant set if $x(0) \in W$ implies $x(t) \in W$ for all $t \geq 0$ [33]. Figure 5b shows V plotted as a function of X_1 and X_2. We note that the function V has a positive value everywhere in $\mathbb{R}^2$ (which includes Ω) except when $H(X_1, X_2) = H^*$, where it is zero. Let us now evaluate $\dot{V}$ along the trajectories of the controlled (constrained) system (5.6) and determine the region in phase space where $\dot{V}$ is guaranteed to be nonpositive.

$$\begin{aligned}
&\dot{V}(X_1, X_2) \\
&\quad = (H - H^*)\,\frac{dH}{dt} \\
&\quad = (H - H^*)\,\frac{d}{dt}\left\{\frac{1}{2}X_2^2 + \frac{a_o}{b_o}e^{b_o X_1} - a_o X_1 - \frac{a_o}{b_o}\right\} \\
&\quad = (H - H^*)\left\{X_2\dot{X}_2 + a_o\dot{X}_1(e^{b_o X_1} - 1)\right\} \\
&\quad = (H - H^*)\left\{\begin{array}{c} -a_o(e^{b_o X_1} - 1)X_2 - \lambda_o\left(H - H^*\right)X_2^2 \\ +a_o(e^{b_o X_1} - 1)X_2 \end{array}\right\} \\
&\quad = -\lambda_o\left(H - H^*\right)^2 X_2^2 \leq 0\ \forall\, \mathbb{R}^2
\end{aligned} \tag{5.9}$$

Thus, we find that $\dot{V}$ is indeed nonpositive throughout $\mathbb{R}^2$ (which includes the set Ω). Since $V \geq 0$ (5.7) and $\dot{V} \leq 0$ (5.9) at all points that lie in the set Ω, we deduce that the trajectories that enter the set Ω at $t = 0$ are confined to it for all future time. This implies that Ω is positively invariant.

3. *Set E* The set E is defined as consisting of all points in Ω where $\dot{V} = 0$. From Eq. (5.9), we deduce that $\dot{V}$ is zero in the set Ω when

$$E = \left\{(X_1, X_2) \in \mathbb{R}^2 \mid X_2 \equiv 0 \cup H(X_1, X_2) \equiv H^*\right\}. \tag{5.10}$$

4. *Set P* The set P is defined to be the union of all invariant sets within E [34]. The set of all points satisfying $H(X_1, X_2) = H^*$ is positively invariant because when $H(X_1, X_2) = H^*$ is substituted into the constrained equations of motion (5.6), the control force is zero, and we obtain our unconstrained (uncontrolled) system which is conservative and for which the energy remains constant (which in this case is H^*) for all time t. On the other hand, substituting $X_2 \equiv 0$ (and therefore $\dot{X}_2 \equiv 0$) in Eq. (5.6), we find that $X_1 \equiv 0$. Thus, the origin ($X_1 \equiv X_2 \equiv 0$) is the only point on the line $X_2 = 0$ which is invariant, and all other trajectories originating on the line $X_2 = 0$ move away from it. However, since the origin (at which $H(0, 0) = 0$) is itself excluded from the set Ω, the set P consists of

$$P = \left\{(X_1, X_2) \in \mathbb{R}^2 \mid H(X_1, X_2) = H^*,\ H^* > 0\right\}. \tag{5.11}$$

Then, by the invariance principle, every solution $x(t)$ starting in Ω approaches P as $t \to \infty$. Thus, we conclude that all trajectories in the set Ω have to eventually converge to the limit cycle (5.11) as time t tends to infinity. Hence, global asymptotic convergence to the limit cycle has been established in Ω. Now, since c (outer boundary of the set Ω) can be chosen to be arbitrarily large and ε (inner boundary of the set Ω encircling the origin) can be chosen to be arbitrarily small, all orbits in $\mathbb{R}^2 - \{\mathrm{O}\}$ asymptotically tend to P. Furthermore, since the open region around the origin can be made arbitrarily small through a proper choice of ε, the origin is an unstable fixed point. This proves that the control force derived in (5.3) for the case of an SDOF Toda oscillator (with initial energy $H_o > 0$) gives us global asymptotic convergence to any given desired energy state H^* in $\mathbb{R}^2 - \{\mathrm{O}\}$. We next proceed to the NDOF system.

5.2 NDOF Toda lattice with fixed–fixed (and fixed–free) boundary conditions

In this section, our aim is to show that for a Toda lattice with initial energy $H_o > 0$:

1. The control force F^C (Eq. 4.12) gives us global asymptotic convergence to any given nonzero desired energy state H^* provided that the first mass, or the last mass, or alternatively any two consecutive masses of the NDOF Toda lattice are included in the subset of masses that are controlled.
2. The controlled NDOF system (5.12) possesses a single isolated equilibrium point at the origin O of the phase space. This is proved in Appendix 3. We show that this fixed point at the origin is unstable.

LaSalle's invariance principle helps us in establishing both these results.

The constrained (controlled) equations of motion of a nonhomogeneous NDOF Toda lattice with fixed–fixed (or fixed–free) boundary conditions can be written as

$$M\ddot{q} = F - \lambda_o \left(H(q,\dot{q}) - H^*\right) I_C M\dot{q}. \tag{5.12}$$

Similar to the SDOF system, let us consider a continuously differentiable scalar function V as

$$V(q,\dot{q}) = \frac{1}{2}\left(H(q,\dot{q}) - H^*\right)^2, \quad \text{where} \quad H^* > 0, \tag{5.13}$$

defined on the set Ω described by

$$\Omega = \left\{(q,\dot{q}) \in \mathbb{R}^{2n} \,|\, \varepsilon \le H(q,\dot{q}) \le c\right\}, \tag{5.14}$$

where $0 < \varepsilon < H^* < c$. By choosing $\varepsilon > 0$, an open region around the origin O (prescribed by $q \equiv \dot{q} \equiv 0$) is excluded from the set Ω. Our objective in choosing Ω as in (5.14) is to establish that the origin O is an unstable fixed point, and all trajectories in $\mathbb{R}^{2n} - \{\text{O}\}$ asymptotically converge to the compact and invariant set defined by $H(q,\dot{q}) = H^*$. Now, to apply the invariance principle, we need to first establish that the set Ω is compact and positively invariant in $2n$-dimensional phase space.

1. **Ω** *is a compact set* A detailed derivation of this result is presented in Appendix 4.

2. **Ω** *is positively invariant* Let us compute $\dot{V}$ along the trajectories of the constrained NDOF Toda lattice (5.12) as shown below.

$$\begin{aligned}
\dot{V}(q,\dot{q}) &= \left(H - H^*\right)\frac{\mathrm{d}H}{\mathrm{d}t}\\
&= \left(H - H^*\right)\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\dot{q}^T M\dot{q} + U(q)\right)\right]\\
&= \left(H - H^*\right)\left[\dot{q}^T\left(M\ddot{q}\right) + \dot{q}^T(-F)\right]\\
&= \left(H - H^*\right)\left[\dot{q}^T\left(F - \lambda_o\left(H - H^*\right)C^T C\, M\,\dot{q}\right) + \dot{q}^T(-F)\right]\\
&= -\lambda_o\left(H - H^*\right)^2 \dot{q}^T C^T C\, M\,\dot{q}\\
&= -\lambda_o\left(H - H^*\right)^2 \dot{q}_C^T M_C \dot{q}_C \le 0 \;\forall\; \mathbb{R}^{2n}
\end{aligned} \tag{5.15}$$

Since $V \ge 0$ (Eq. 5.13) and $\dot{V} \le 0$ (Eq. 5.15) at all points that lie in the set Ω, we deduce that the set Ω is positively invariant. Note that this result also holds true if β were to be given by Eq. (4.11) instead.

3. *Set E* From Eq. (5.15), we deduce that $\dot{V}$ is zero in the set Ω when

$$E = \left\{(q,\dot{q}) \in \mathbb{R}^{2n} \,|\, \dot{q}_C \equiv 0 \cup H(q,\dot{q}) \equiv H^*\right\}. \tag{5.16}$$

4. *Set P* The set P is defined to be the union of all invariant sets within E [34]. The set of all points satisfying $H(q,\dot{q}) = H^*$ is positively invariant because when $H(q,\dot{q}) = H^*$ is substituted into the equations of motion of the controlled lattice (5.12), the control force is zero, and we obtain our uncontrolled (unconstrained) system (2.7) which is conservative and for which the energy remains constant (which in this case is H^*) for all time t. Next, we need to ensure that the only invariant set in $E(\subseteq \Omega)$ is the set defined by $H(q,\dot{q}) = H^*$, so that all trajectories in Ω are globally attracted to this set. Thus, we would ideally like the invariant set(s) satisfying $\dot{q}_C \equiv 0$ to lie outside Ω. To ensure this, we need to place the actuators appropriately, so that

$$\dot{q}_C \equiv 0 \quad \text{only yields} \quad q \equiv \dot{q} \equiv 0, \tag{5.17}$$

which is invariant, and which does not belong to the set Ω. A sufficient condition for (5.17) to occur in NDOF Toda lattices with fixed–fixed (or fixed–free) ends is when the set of locations of the actuators includes at least one of the following configurations (see Appendix 5 for a detailed derivation).

1. A single actuator is placed on the first mass and/or the last mass of the lattice.
2. Two actuators are placed on two consecutive masses located anywhere in the lattice.

Table 1 Description of the four numerical simulation examples considered in this section

Example number	Boundary conditions	Homogeneity of lattice	Location of initial excitation	Energy raised/ lowered	Actuator locations	λ_o
1	Fixed–Fixed	Homogeneous and Nonhomogeneous	m_{51}	Raised	m_{75}, m_{76}	0.1
2	Fixed–Fixed	Nonhomogeneous	m_{51}	Raised	Multiple sets of actuator configurations	0.01
3	Fixed–Free	Homogeneous and Nonhomogeneous	m_{51}	Lowered	m_{75}, m_{76}	0.1
4	Fixed–Free	Nonhomogeneous	Random initial excitation (all masses)	Lowered	m_{101}	0.1

Now, since the origin O ($q \equiv \dot{q} \equiv 0$) is excluded from the set Ω, the largest invariant set in E is given by

$$P = \left\{ (q, \dot{q}) \in \mathbb{R}^{2n} \mid H(q, \dot{q}) = H^*;\ H^* > 0 \right\}. \tag{5.18}$$

Then, by the invariance principle, every solution $x(t)$ starting in Ω approaches P as $t \rightarrow \infty$. Thus, global asymptotic convergence to the set $H(q, \dot{q}) = H^*$ has been established in Ω. Now, since c can be chosen to be arbitrarily large and ε can be chosen to be arbitrarily small, all orbits in $\mathbb{R}^{2n} - \{\mathrm{O}\}$ asymptotically tend to P. Moreover, since the open region around the origin can be made arbitrarily small through a proper choice of ε, the origin is an unstable fixed point.

This proves then that for a nonhomogeneous NDOF Toda lattice with initial energy $H_o > 0$, the control force F^C derived in (4.12) gives us global asymptotic convergence to any given desired energy state H^* in $\mathbb{R}^{2n} - \{\mathrm{O}\}$ provided that the first mass, or the last mass, or alternatively any two consecutive masses of the Toda lattice are included in the subset of masses that are controlled. The result is valid for both fixed–fixed and fixed–free boundary conditions.

6 Results and simulations

In this section, numerical simulations involving a Toda lattice with fixed–fixed and fixed–free boundary conditions are presented to illustrate the ease and efficacy with which the control methods described in this paper can be applied. Since n can be any finitely large number, a 101-mass lattice is chosen.

One of the significant results of this paper is that though the control is highly underactuated, it is still guaranteed to control the energy of the lattice from any nonzero initial state to any other nonzero desired final state. As shown analytically in Sect. 5.2, to achieve this desired energy state, one could use just a single actuator placed on the first mass m_1 of the lattice (or on the last mass m_n of the of the lattice, see Fig. 1), or one can simply actuate two neighboring masses located anywhere in the lattice, no matter how many degrees of freedom the lattice has. This is shown to be true for different boundary conditions, and whether or not the lattice is homogeneous. Furthermore, the rate of convergence to the desired energy state can also be controlled, and in addition, various sets of actuator locations can be chosen. To illustrate the scope of all these qualitatively different results, four different examples are considered as shown in Table 1. They are described below.

Boundary conditions and homogeneity of the lattice Examples 1 and 2 deal with a fixed–fixed Toda lattice, whereas Examples 3 and 4 deal with a fixed–free Toda lattice. Examples 1 and 3 consider both a homogeneous lattice (for purposes of comparison) and a nonhomogeneous lattice. Examples 2 and 4 deal exclusively with nonhomogeneous lattices.

Specification of the lattice parameters In Example 1, the values of the spring constants (a_i, b_i) and the masses (m_i) of the fixed–fixed homogeneous lattice (see Fig. 1) are taken to be $a_i = 2$, $b_i = 1$ for $0 \leq i \leq 101$, and $m_i = 1$ for $1 \leq i \leq 101$. For the fixed–free homogeneous Toda lattice in Example 3, the last spring element (i.e., the spring element with spring constants a_{101}, b_{101}) is discarded keeping all the other spring element and mass parameter values the same as in Example 1. The values of the spring constants and the masses of the fixed–fixed nonhomogeneous lattice in Example 1 are selected at random from a uniformly distributed set of numbers between the limits: $1.5 < a_i < 2.5$, $0.5 < b_i < 1.5$ for $0 \leq i \leq 101$, $0.5 < m_i < 1.5$ for $1 \leq i \leq 101$. The nonhomogeneous lattice in Example 2 has the same parameter values as those in Example 1. The parameter values of the fixed–free nonhomogeneous lattices in Examples 3 and 4 are also the same as those in Example 1, except that the last spring element is discarded, as before.

Specification of initial conditions In Examples 1, 2, and 3, the lattice is initially excited with all masses having zero initial displacement and zero initial velocity except for the mass located at the center of the lattice, m_{51}, which is given an initial displacement. In Example 4, all the masses of the fixed–free lattice are excited with random initial displacements and random initial velocities.

Specification of energy control requirements In all of the examples, the aim is to control the energy in these respective lattices and bring them to the desired energy level. In Examples 1 and 2, the energy of the lattice is desired to be increased from its initial state, while in the latter two examples, the energy is desired to be reduced.

Actuator locations To achieve these desired energy levels, control can be applied to one or more of these 101 masses (provided that the first mass, or the last mass, or alternatively any two consecutive masses of the lattice are included in the subset of masses that are controlled). In Examples 1 and 3, control is applied to two consecutive masses, m_{75} and m_{76}, located at about three-quarters the distance from the left end of the lattice (see Fig. 1). In Example 2, five different sets of actuator configurations are chosen to exhibit the effect of the placement of the actuators on the time it takes for the controlled lattice to get to the desired energy level. In Example 4, only the last mass of the lattice is actuated in order to lower its energy.

The constant λ_o (see Eq. 4.12) which affects the rate at which the controlled lattice converges to the desired energy level is chosen to be 0.1 in all of the examples, except in Example 2 where it is lowered to 0.01 to allow for an easier comparison of the times taken by the different sets of actuator configurations to reach the desired energy level. In all four examples considered in this section, the equations of motion are integrated using *ode113* in the MATLAB environment with a relative integration error tolerance of 1e–10 and an absolute error tolerance of 1e–13. All quantities are assumed to be in consistent units.

6.1 Fixed–fixed Toda lattice

Example 1 In the first example, we study a 101-mass Toda lattice with fixed–fixed boundary conditions. Our aim is to raise the energy of both a homogeneous lattice (for comparison) and a nonhomogeneous lattice (both of whose parameters are as described earlier) to a desired level of 150 units in each case. The mass located at the center of the lattice, m_{51}, is initially displaced by 3 units, both in the homogeneous and the nonhomogeneous lattice. This causes the initial energy level, H_o, of the homogeneous and the nonhomogeneous lattice to be 36.27 and 83.58 units, respectively.

Homogeneous lattice Figure 6a shows the velocity field for the uncontrolled (unconstrained) homogeneous lattice, where time is plotted on the x-axis, and the location of the masses is plotted on the y-axis. The velocity of each mass, which is plotted on the z-axis, is instead shown through a color variation (see color scale on the right of the figure). As shown in Fig. 6a, the initial displacement of the mass at the center of the lattice gives rise to, what appear to be, multiple soliton structures propagating through the velocity field. Among many small waves, there appear to be two large solitons generated at $t = 0$, one with positive velocity amplitude (shown in dark red) traveling toward the left end of the lattice (toward mass m_1) and another with negative velocity amplitude (shown in dark blue) traveling toward the right end of the lattice (toward mass m_{101}). When each of these solitons reaches the fixed end, the incident soliton is reflected, and the reflected soliton has its amplitude reversed in sign. The propagation speed of these individual solitons appears to be constant (as seen from the slopes of these lines), with the two large solitons traveling at a faster rate than the smaller waves. The reflections of these two large solitons first cross each other at around 52 s, and they continue to propagate undisturbed after they cross.

The explicit closed-form control given in Eq. (4.12) is now applied to this homogeneous lattice at masses

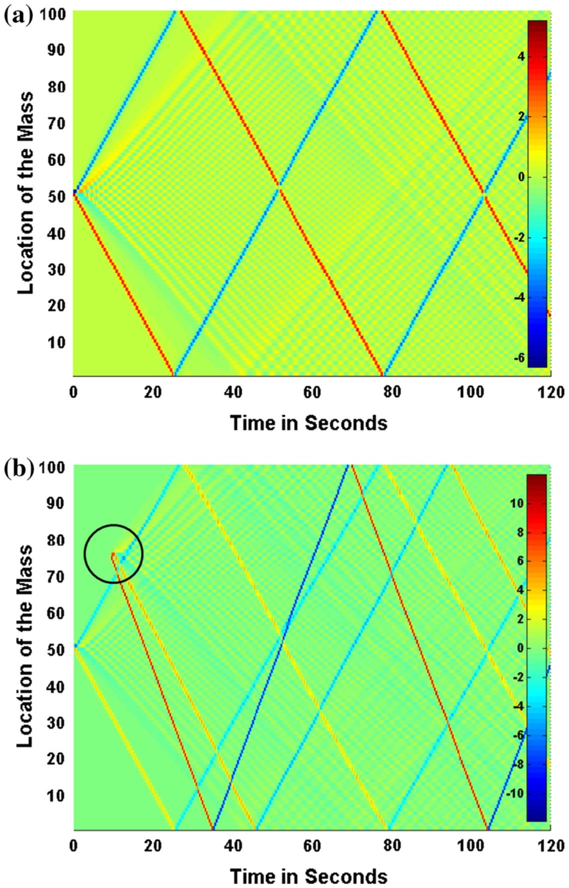

Fig. 6 Example 1: Velocity field of the 101-mass fixed–fixed homogeneous Toda lattice. The lattice has parameters $a_i = 2$, $b_i = 1$, $m_i = 1$ for all i, $H_o = 36.27$, $H^* = 150$, $\lambda_o = 0.1$ and the initial displacement of center mass (m_{51}) is 3 units. Actuators are located on masses m_{75} and m_{76}. **a** Uncontrolled homogeneous lattice. **b** Controlled homogeneous lattice. (Color figure online)

m_{75} and m_{76}, so that the controlled lattice achieves the desired energy level of 150 units. A plot of the control forces is shown in Fig. 8a from $t = 0$ to $t = 25$ s. The dotted red and blue lines denote the control forces acting on actuator masses m_{75} and m_{76}, respectively. As shown in the plot, a finite amount of time elapses before the control begins. This is because it takes a finite amount of time for the initial excitation at the center of the lattice to traverse through the lattice and reach the actuator locations at m_{75} and m_{76}. Since the control forces depend on the velocity of the actuated masses [see Eq. (4.12)], once the actuator masses are in motion at around 9 s, the control begins and the desired energy state of $H^* = 150$ (see Fig. 8b, top) is very quickly achieved. Once the desired energy state is achieved, the control forces terminate (see Fig. 8a) and the conservative nature of the lattice is utilized to maintain its energy at the desired level for all future time. Figure 6b shows a plot of the velocity field for the controlled homogeneous lattice. From the figure, we observe that coinciding with the application of the control forces, a new set of soliton structures is generated (see black circle in the figure) in addition to those already generated by the initial displacement of the center mass. Some of these newly generated solitons have larger amplitudes and higher propagation speeds when compared to those extant. Figure 8b (bottom subplot) shows the energy error, $e(t) = H(t) - H^*$, in achieving the desired energy level plotted as a function of time from $t = 400$ to $t = 500$ s. As shown in the figure, the magnitude of this error is commensurate with the error tolerances used in integrating the equations of motion of the controlled system.

Nonhomogeneous lattice We next consider the uncontrolled nonhomogeneous lattice whose velocity field is shown in Fig. 7a. As shown in the figure, the initial displacement of the mass at the center of the lattice, m_{51}, generates waves at $t = 0$, which traverse through the length of the lattice. The crisscross pattern shown in the figure is generated by the propagation of these waves and their reflection at the boundaries. The distinct soliton structures that were observed in the velocity field of the homogeneous lattice are no longer present in the nonhomogeneous lattice, and this absence is due to the nonhomogeneity of the lattice. Once again, we apply control [see Eq. (4.12)] to this nonhomogeneous lattice to raise its energy level from $H_o = 83.58$ units to $H^* = 150$ units. The time history of the control forces obtained by using Eq. (4.12) is shown in Fig. 8a where the solid red and blue lines denote the control forces acting on the actuator masses m_{75} and m_{76}, respectively. The control begins at around 11 s and again quickly stabilizes the lattice at the desired energy level (see Fig. 8b, top). The control generates its own velocity field causing waves to emanate at around 11 s (see black circle in Fig. 7b) in addition to those already generated by the initial displacement of the center mass. Figure 8b (bottom subplot) shows the energy error, $e(t) = H(t) - H^*$, as before.

Example 2 In this example, a 101-mass nonhomogeneous Toda lattice with fixed–fixed ends is considered with the same parameter values for the spring elements

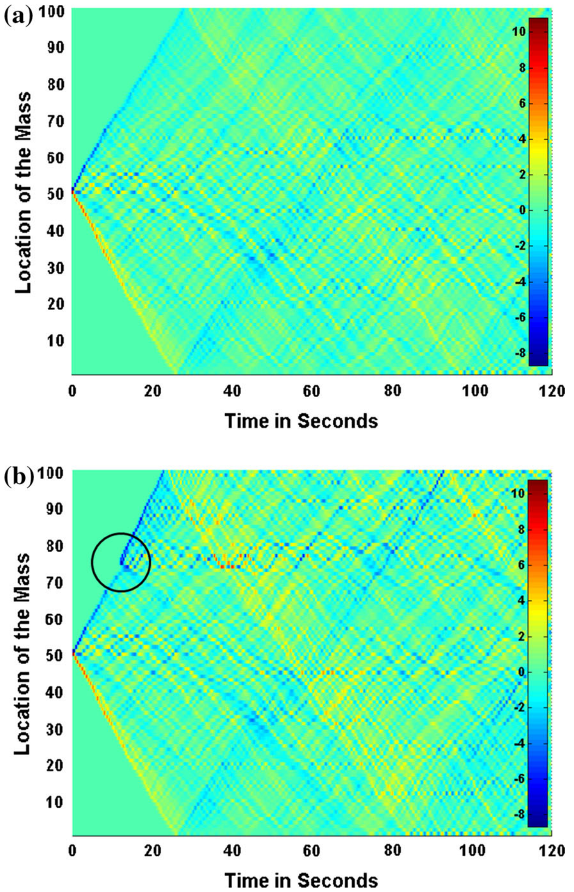

Fig. 7 Example 1: Velocity field of the 101-mass fixed–fixed nonhomogeneous Toda lattice. The nonhomogeneous lattice has parameters a, b, and m chosen randomly from a uniformly distributed set of numbers between the limits: $1.5 < a_i < 2.5$, $0.5 < b_i < 1.5$ and $0.5 < m_i < 1.5$, $H_o = 83.58$, $H^* = 150$, $\lambda_o = 0.1$, and the initial displacement of center mass (m_{51}) is 3 units. Actuators are located on masses m_{75} and m_{76}. **a** Uncontrolled nonhomogeneous lattice. **b** Controlled nonhomogeneous lattice. (Color figure online)

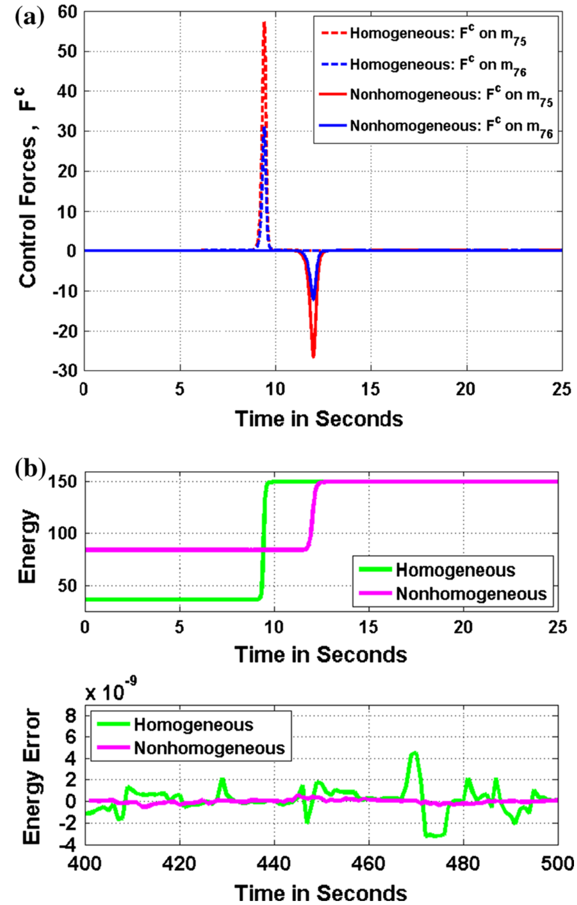

Fig. 8 **a** Example 1: Time history of control forces acting on the two actuator masses m_{75} and m_{76} of the lattice. Dotted lines denote control forces acting on the homogeneous lattice, and the solid lines denote the control forces acting on the nonhomogeneous lattice. Red line denotes the control force acting on the mass m_{75}, and blue line denotes the control force acting on mass m_{76}. **b** Example 1: (Top subplot) Time history of energy of the lattice from $t = 0$ to 25 s. (Bottom subplot) Time history of energy error $e(t) = H(t) - H^*$ from $t = 400$ to 500 s. Green line denotes the homogeneous lattice and pink line denotes the nonhomogeneous lattice. (Color figure online)

and the masses as in the previous example. An initial displacement of 3 units is given to the mass located at the center of the lattice like in the previous example, and this causes the initial energy of the lattice to be $H_o = 83.58$ units. Five different sets of actuator configurations are considered (in accordance with the actuator location rules specified in Appendix 5) to study their effect on the time it takes for the controlled nonhomogeneous lattice to reach a desired energy level of 150 units. The configurations are (see Fig. 9a): (1) only mass m_1 is actuated; (2) only masses m_{75} and m_{76} are actuated; (3) only masses m_{50} and m_{51} are actuated; (4) every tenth mass along the lattice is actuated starting with mass m_1; and (5) every fifth mass along the lattice is actuated starting with mass m_1. As described earlier, the parameter λ_o (see Eq. 4.12) is lowered to 0.01 in the present example.

Figure 9 shows a plot of the convergence of the lattice's energy to the desired energy level as a function of time for the five sets of actuator configurations that are considered. From the figure, we observe that the lattice controlled using two consecutive actuators placed at roughly three-quarters the distance from the left end of the lattice at m_{75}, and m_{76} takes the longest time to

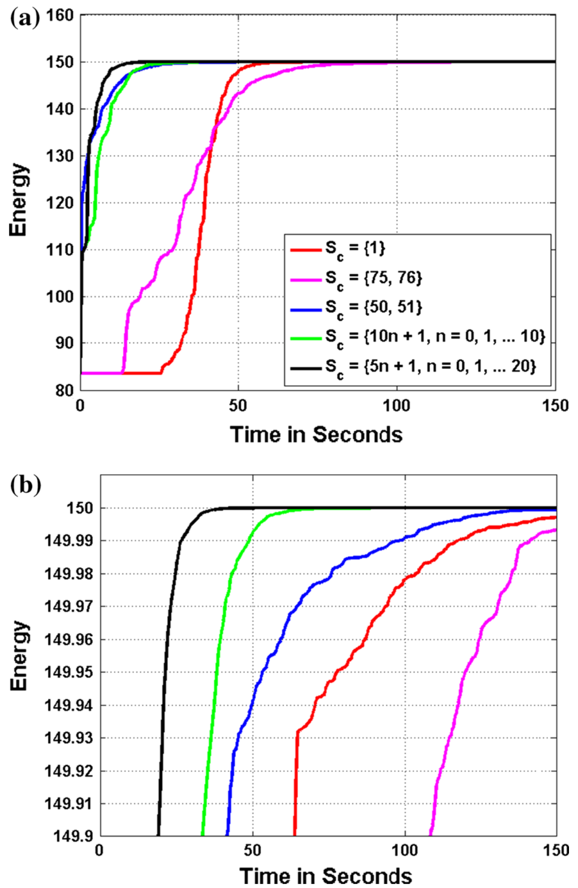

Fig. 9 Example 2: Nonhomogeneous Toda lattice with fixed–fixed boundary conditions. The lattice has its parameters chosen randomly from a uniformly distributed set of numbers between the limits: $1.5 < a_i < 2.5, 0.5 < b_i < 1.5, 0.5 < m_i < 1.5$, $H_o = 83.58$, $H^* = 150$, $\lambda_o = 0.01$, and initial displacement of the center mass (m_{51}) is 3 units. **a** Time history of energy convergence for different sets of actuator configurations of the 101-mass fixed–fixed nonhomogeneous Toda lattice. **b** Zoomed-in plot showing the energy convergence for the five different sets of actuator configurations. (Color figure online)

reach the desired energy state, longer even compared to a single actuator placed on the first mass (m_1) of the lattice. However, if one chooses to place these actuators on two consecutive masses at m_{50} and m_{51}, closer to the initial excitation at the center of the lattice (i.e., at mass m_{51}), the desired energy state is achieved comparatively faster (in approximately 140 s). Furthermore, using 11 and 21 equidistantly placed actuators with the first of these actuators placed on the first mass of the lattice helps us in attaining the energy state in approximately 80 s and 45 s, respectively. We, thus, conclude that the time taken for the controlled lattice to reach the desired energy state depends on the number of actuators, the placement of these actuators, as well as on the nature of the initial excitation of the lattice. Though an interesting problem in itself, we, however, do not delve deeper into it here as it will take us too far afield from the central focus of this paper.

6.2 Fixed–free Toda lattice

Example 3 In this example, we study a 101-mass Toda lattice with fixed–free boundary conditions. Our aim is to lower the energy of a homogeneous lattice (for comparison) and a nonhomogeneous lattice from an initial energy level of approximately 150 units to a desired level of 100 units. The parameter values of the lattices are as described earlier. To initialize both lattices at an energy level of approximately 150 units, the initial displacement of the mass at the center of the lattice, m_{51}, is chosen to be 4.34 units in the homogeneous case (this causes its initial energy level, H_o, to be 149.44 units) and 3.41 units in the nonhomogeneous case (initial energy level, H_o, is 149.90 units), with all the other masses in the respective lattices given zero initial displacement and zero initial velocity.

Homogeneous lattice A plot of the velocity field of the homogeneous lattice is shown in Fig. 10. Once again, similar to Example 1, we observe that the initial displacement of the center mass gives rise to two large solitons in addition to many small waves in the velocity field of the uncontrolled homogeneous lattice (see Fig. 10a). Since the left end of the lattice is a fixed end, the positive velocity amplitude soliton traveling toward m_1 is reflected, and the reflected soliton has its velocity amplitude reversed in sign, but its magnitude remains the same as the incident soliton. On the other hand, the large negative amplitude soliton traveling toward the free end of the lattice toward m_{101} is imperfectly reflected. The free end has the effect of breaking up the incident soliton into a smaller amplitude (reflected) soliton and many small (reflected) waves that can be seen traversing through the velocity field (see Fig. 10a). This behavior is consistent with what has been documented in the literature [35]. Next, we apply control to this homogeneous lattice at masses m_{75} and m_{76} to reduce its energy level to 100 units. From Fig. 10b, we observe that the application of the control force breaks

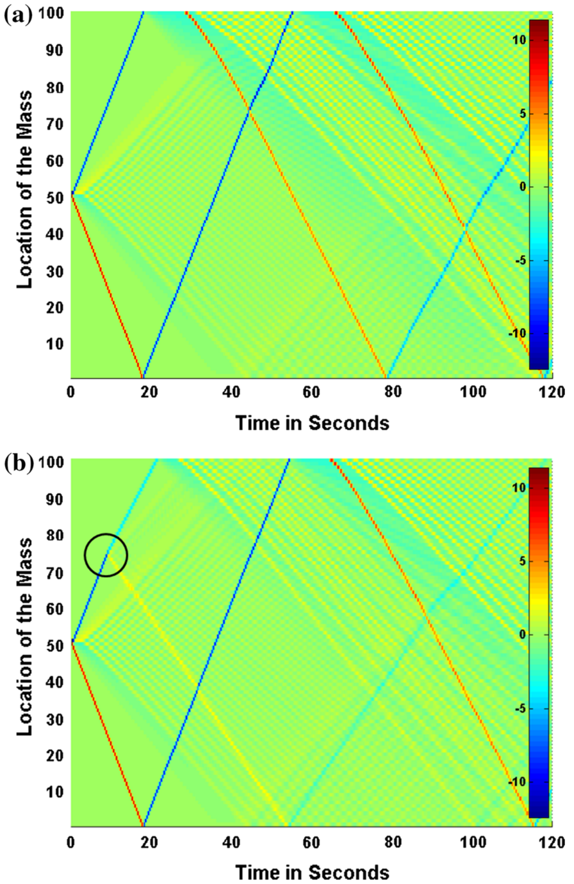

Fig. 10 Example 3: Velocity field of the 101-mass fixed–free homogeneous Toda lattice. The lattice has parameters $a_i = 2$, $b_i = 1$, $m_i = 1$ for all i, $H_o = 149.44$, $H^* = 100$, $\lambda_o = 0.1$ and the initial displacement of center mass (m_{51}) is 4.34 units. Actuators are located on masses m_{75} and m_{76}. **a** Uncontrolled homogeneous lattice. **b** Controlled homogeneous lattice. (Color figure online)

the large negative velocity amplitude soliton (generated by the initial displacement of the center mass m_{51}) into two slower-moving smaller solitons (see black circle in the figure), one having positive velocity amplitude traveling toward the left end of the lattice and another having negative amplitude traveling toward the right end of the lattice. The time history of control forces acting on the homogeneous lattice is shown in the top subplot of Fig. 12a. Figure 12b shows that the desired energy level of 100 units is achieved, and the energy error lies close to the tolerance levels specified in the integration algorithm.

Nonhomogeneous lattice Figure 11a shows a plot of the velocity field of the uncontrolled nonhomogeneous lattice with fixed–free ends, wherein the mass at the center of the lattice, m_{51}, is provided an initial displacement of 3.41 units. Once again, like in Example 1, the absence of distinct soliton structures in the velocity field of the nonhomogeneous lattice is evident. Control is applied to masses m_{75} and m_{76} to reduce the lattice's energy from 149.9 to 100 units. From the velocity field of the controlled lattice (shown in Fig. 11b), we note that the set of masses located beyond masses m_{75} and m_{76} have amplitudes of motion that are smaller than the other masses in the lattice. The actuators at m_{75} and m_{76} damp out the motion in the lattice through destructive interference in order to reduce the lattice's energy.

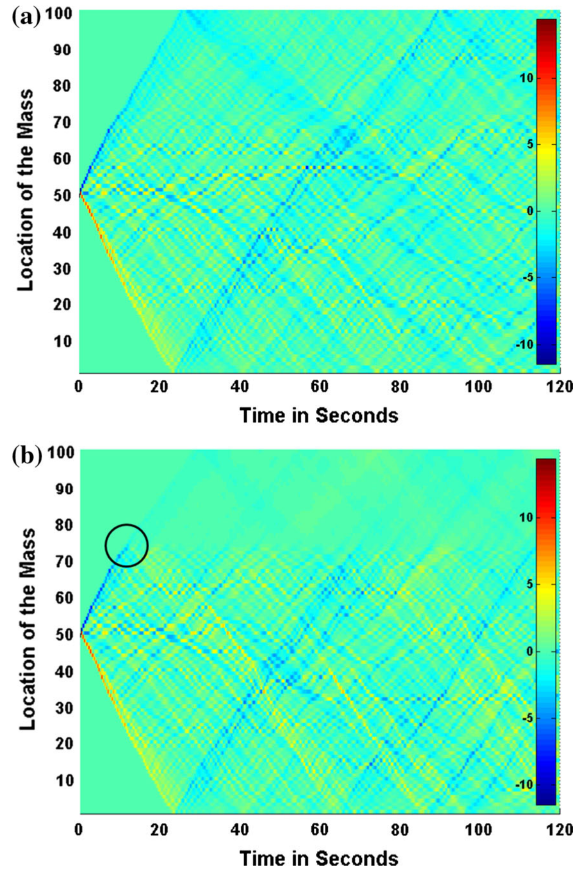

Fig. 11 Example 3: Velocity field of the 101-mass fixed–free nonhomogeneous Toda lattice. The nonhomogeneous lattice has parameters a, b, and m chosen randomly from a uniformly distributed set of numbers between the limits: $1.5 < a_i < 2.5$, $0.5 < b_i < 1.5$ and $0.5 < m_i < 1.5$, $H_o = 149.9$, $H^* = 100$, $\lambda_o = 0.1$, and the initial displacement of center mass (m_{51}) is 3.41 units. Actuators are located on masses m_{75} and m_{76}. **a** Uncontrolled nonhomogeneous lattice. **b** Controlled nonhomogeneous lattice. (Color figure online)

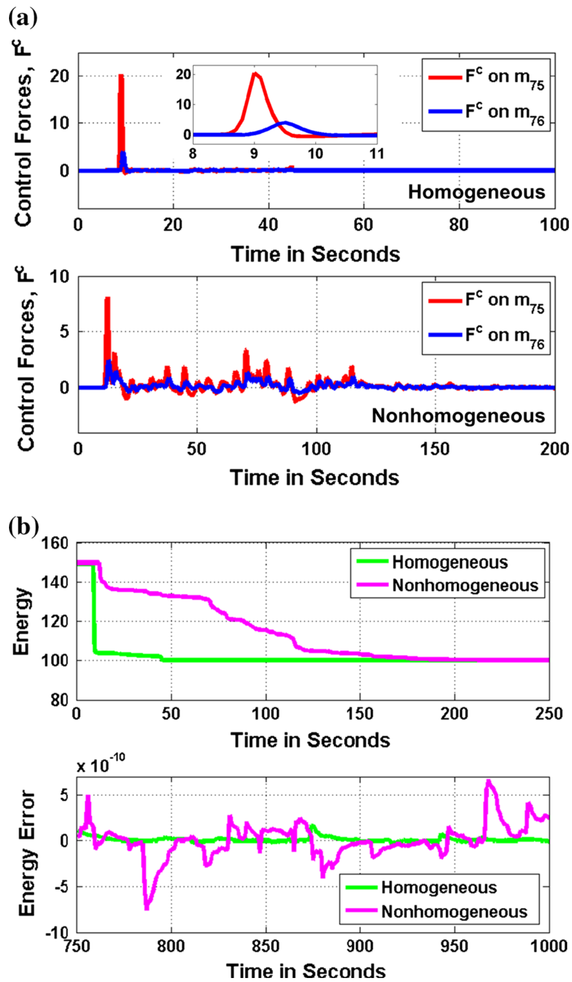

Fig. 12 **a** Example 3: Time history of control forces acting on the two actuator masses m_{75} (red line) and m_{76} (blue line) of the homogeneous lattice (top subplot) and the nonhomogeneous lattice (bottom subplot). **b** Example 3: (Top subplot) Time history of energy from $t = 0$ to 250 s. (Bottom subplot) Time history of energy error $e(t) = H(t) - H^*$ from $t = 750$ to 1000 s. Green line denotes the homogeneous case and pink denotes the nonhomogeneous case. (Color figure online)

Comparing Figs. 12(a) to 8(a), we note that the control acts for a longer duration, and it takes a longer time for the lattice to reach the desired energy state (both in the homogeneous and nonhomogeneous lattice) when the energy of the lattice is desired to be reduced as compared to when the energy of the lattice is desired to be raised. As before, Fig. 12b shows that the desired level of 100 units is attained, and that the energy errors lie close to the tolerance levels specified by the integration algorithm.

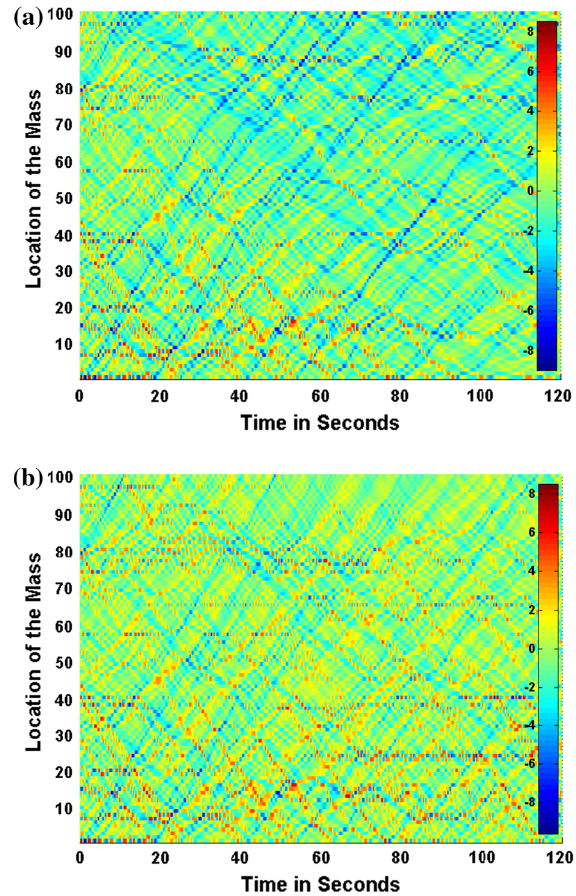

Fig. 13 Example 4: Velocity field of the 101-mass fixed–free nonhomogeneous Toda lattice subjected to random initial excitation. The nonhomogeneous lattice has parameters a, b, and m chosen randomly from a uniformly distributed set of numbers between the limits: $1.5 < a_i < 2.5, 0.5 < b_i < 1.5$ and $0.5 < m_i < 1.5$, $H_o = 341.47$, $H^* = 250$, $\lambda_o = 0.1$. The initial displacements and initial velocities are chosen randomly between the limits -2 and 2. A single actuator is placed on the last mass of the fixed–free lattice (i.e., on m_{101}). **a** Uncontrolled nonhomogeneous lattice. **b** Controlled nonhomogeneous lattice. (Color figure online)

Example 4 In this final example, a fixed–free nonhomogeneous Toda lattice subjected to random initial excitation is considered. The initial displacements and initial velocities of each of the 101 masses in the lattice are chosen at random from a uniformly distributed set of numbers between the limits -2 and 2. This random selection of the initial conditions causes the lattice to have an initial energy level $H_o = 341.47$ units, and the aim is to reduce this energy to $H^* = 250$ units using only a single actuator located on the last mass (m_{101}) of the fixed–free lattice.

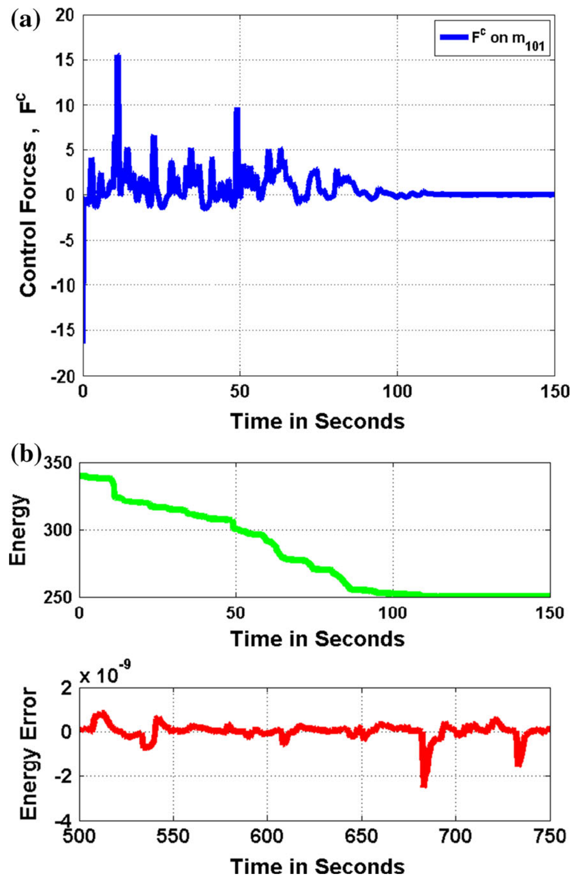

Fig. 14 **a** Example 4: Time history of control forces acting on the last mass m_{101} of the nonhomogeneous lattice. **b** Example 4: (Top subplot) Time history of energy from $t = 0$ to 150 s. (Bottom subplot) Time history of energy error $e(t) = H(t) - H^*$ from $t = 500$ to 750 s. (Color figure online)

Figure 13a shows the velocity field of the uncontrolled nonhomogeneous lattice subjected to random initial conditions. Control is now applied to the last mass (m_{101}) of the fixed–free lattice to reduce its energy. Figure 13b shows the velocity field of the controlled nonhomogeneous lattice, where we observe that the response of the controlled lattice is markedly different from the uncontrolled lattice. This is especially evident near right end of the lattice (close to the last mass m_{101}) where the motion of the lattice is being damped out through destructive interference in order to reduce its energy. Figure 14a shows a time history of the control forces acting on the last mass of the lattice where we note that the control acts right from the instant $t = 0$. Figure 14b shows a plot of the energy convergence and energy errors in achieving the desired energy state, as before.

In all of the examples considered in this section, once the desired energy state is achieved, the control forces automatically become zero (see Figs. 8a, 12a, 14a), and the conservative nature of the lattice is thereafter utilized to maintain its energy at the desired level for all future time. Further, in all of the examples, the energy errors are small and lie close to the tolerance levels specified in our integration algorithm (see Figs. 8b, 12b, 14b), highlighting the efficacy of the control methodology in achieving the desired energy state.

7 Conclusions

This paper considers the energy control problem of an n-degrees-of-freedom nonhomogeneous Toda lattice with fixed–fixed (and fixed–free) boundary conditions. Unlike previous investigators, we consider a nonhomogeneous Toda lattice, which has more practical applicability as opposed to the idealized homogenous case. The control approach adopted in this paper is inspired by recent results in analytical dynamics that deal with the theory of constrained motion. For a given set of masses at which the control is to be applied, explicit *closed-form* expressions for the nonlinear control forces are obtained by using the fundamental equation of mechanics. In spite of the nonhomogeneous nature of the Toda lattice considered in this study, the control is obtained with relative ease and, in closed form, without the need for any approximations and/or linearizations of the nonlinear dynamical system and without the need to impose any a priori structure on the nature of the nonlinear controller. The resulting equations of motion of the controlled Toda lattice resemble those of a self-excited system akin to a Van der Pol nonlinear oscillator. The control forces act on the NDOF Toda lattice to bring it to the desired energy level; once this energy level is attained, the control forces automatically terminate, and the conservative nature of the lattice is thereafter utilized to maintain its energy at the desired level for all future time. The control forces, F^C, are continuous in time and are optimal; they minimize the control cost given by $J(t) = [F^C]^T M^{-1} [F^C]$ at each instant of time while causing the energy constraint (Eq. 4.6) to be exactly satisfied. LaSalle's invariance principle is used to establish that the control forces give us global asymptotic convergence to any given nonzero desired energy state provided that the first mass, or the last mass, or alternatively any two consecutive masses of the lattice are included in the subset of masses that are controlled. The manifold $H(q, \dot{q}) = H^*$ forms a globally attracting limit surface in $2n$-dimensional phase space, and the trajectories of the controlled system asymptotically tend to this surface. The Toda lattice is highly underactuated. With just one actuator placed at either end of the chain, or with two actuators placed adjacent to each other anywhere in the chain, the energy of the entire lattice can be controlled. Numerical simulations involving a 101-mass Toda lattice with fixed–fixed and fixed–free boundary conditions are presented, where the behavior of homogeneous and nonhomogeneous lattices is contrasted. Complex nonlinear wave interactions are found in both the uncontrolled and the controlled systems. The closed-form control obtained analytically is shown to work well for both increasing and decreasing the energy of the nonlinear lattice demonstrating the ease, simplicity, and accuracy with which the general control methodology works. The interested reader can refer to Refs. [27–31,36,37] for other complex nonlinear systems in which this methodology has been shown to work well.

Appendix 1: H is a positive definite function in 2 n-dimensional phase space

The energy function H for a nonhomogeneous NDOF Toda lattice is given by

$$H(x) = H(q, \dot{q}) = T(\dot{q}) + U(q) = \sum_{i=1}^{n+1} \left[\frac{1}{2} m_i \dot{q}_i^2 \right] + \sum_{i=0}^{n} \left[\frac{a_i}{b_i} e^{b_i (q_{i+1} - q_i)} - a_i (q_{i+1} - q_i) - \frac{a_i}{b_i} \right]. \quad (7.1)$$

The kinetic energy T is a sum of n quadratic terms with $T(0) = 0$ and $T(\dot{q}) > 0\ \forall\ \dot{q} \neq 0$. The potential energy U on the other hand is a sum of $(n+1)$ functions, $u_i,\ i = 0,\ 1,\ 2,\ \ldots n$, for the fixed–fixed lattice, and a sum of n functions, $u_i, i = 0,\ 1,\ 2,\ \ldots\ n-1$, for the fixed–free lattice ($u_n = 0$ as $a_n = b_n = 0$), where each u_i assumes the form

$$u_i (\delta_i) = \frac{a_i}{b_i} e^{b_i \delta_i} - a_i \delta_i - \frac{a_i}{b_i}, \quad (7.2)$$

where $\delta_i = q_{i+1} - q_i$. For $a_i, b_i > 0$, each potential function u_i is positive definite (see Sect. 2 for reasoning) and hence $u_i(0) = 0$ and $u_i(\delta_i) > 0\ \forall\ \delta_i \neq 0$. Therefore, $U(q) = \sum u_i(\delta_i) > 0\ \forall\ q \neq 0$ (as $q \neq 0$ implies that at least one of the δ_i's is not equal to zero). Further, since the left end is fixed ($q_o \equiv 0$) for both a fixed–fixed and a fixed–free Toda lattice, $\delta_i = 0\ \forall\ i$ implies $q = 0$. Thus, $U(0) = \sum u_i(0) = 0$. Therefore, we obtain $H(0) = 0$ and $H(x) > 0\ \forall\ x \neq 0$ where $x = (q, \dot{q}) \in \mathbb{R}^{2n}$. Hence, the energy function H is strictly positive definite.

Appendix 2: Closed-form expression for the control force F^C

In this appendix, we derive a closed-form expression for the explicit nonlinear control force F^C using Eq. (3.7). The constraint matrices A and b are expressed in terms of the "control selection matrix", C, and the "no-control selection matrix", N (see Sect. 4). And therefore, before we compute the control force, let us list some properties of the matrices C and N.

Properties of the matrices C and N

a. $C^T C + N^T N = I_C + I_N = I_n$, where I_n denotes the n-by-n identity matrix.
b. $CC^T = I_k$, where I_k denotes the k-by-k identity matrix.
c. $NN^T = I_{n-k}$, where I_{n-k} denotes the $(n-k)$-by-$(n-k)$ identity matrix.
d. $I_C \Lambda = C^T C \Lambda = \Lambda C^T C = \Lambda I_C$ for all diagonal matrices Λ.
e. $I_N \Lambda = N^T N \Lambda = \Lambda N^T N = \Lambda I_N$ for all diagonal matrices Λ.
f. $NC^T = [\mathrm{O}]_{(n-k) \times k} \Rightarrow NMC^T = [\mathrm{O}]_{(n-k) \times k}$, where [O] denotes the zero matrix.
g. $CN^T = [\mathrm{O}]_{k \times (n-k)} \Rightarrow CMN^T = [\mathrm{O}]_{k \times (n-k)}$, where [O] denotes the zero matrix.
h. $I_C I_N = I_N I_C = [\mathrm{O}]_{n \times n}$, where [O] denotes the zero matrix.
i. $I_C N^T = C^T C N^T = [\mathrm{O}]_{n \times (n-k)}$, where [O] denotes the zero matrix.
j. $I_N C^T = N^T N C^T = [\mathrm{O}]_{n \times k}$, where [O] denotes the zero matrix.
k. $I_C C^T = C^T,\ I_N N^T = N^T$

The computation of the control force, F^C, involves the evaluation of the Moore–Penrose (MP) inverse [16] of the $(n-k+1)$-by-n matrix B given by

$$B = AM^{-1/2} = \begin{bmatrix} \dot{q}^T M \\ NM \end{bmatrix} M^{-1/2} = \begin{bmatrix} \dot{q}^T M^{1/2} \\ NM^{1/2} \end{bmatrix} \quad (7.3)$$

Given any $(n-k+1)$-by-n matrix B, there exists a unique n-by-$(n-k+1)$ matrix B^+, called the Moore–

Penrose inverse of the matrix B, which satisfies the following four conditions [19].

1. $(BB^+)^T = BB^+$, 2. $(B^+B)^T = B^+B$,
3. $BB^+B = B$, 4. $B^+BB^+ = B^+$

For a matrix B given by (7.3), we claim that B^+ is given by

$$B^+ = \left[\begin{array}{c|c} \frac{M^{1/2}C^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}} & M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}} \end{array}\right] \tag{7.4}$$

Assuming that the B^+ given by (7.4) is indeed the correct expression for the MP inverse of B, we show that it satisfies all four conditions of the MP inverse.

(i)
$$BB^+ = \begin{bmatrix} \dot{q}^TM^{1/2} \\ NM^{1/2} \end{bmatrix} \left[\begin{array}{c|c} \frac{M^{1/2}C^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}} & M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}} \end{array}\right]$$
$$= \left[\begin{array}{c|c} \frac{\dot{q}^TMC^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}} & \dot{q}^TN^T - \frac{(\dot{q}^TMC^TC\dot{q})\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}} \\ \hline \frac{(NMC^T)C\dot{q}}{\dot{q}^TC^TCM\dot{q}} & NN^T - \frac{(NMC^T)C\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}} \end{array}\right] = I_{n-k+1} \tag{7.5}$$

By applying Property (d), the (1, 1) block of (7.5) is unity, and the (1, 2) block simplifies to a $(n-k)$-sized zero row vector. The (2, 1) block is a $(n-k)$-sized column vector which is zero by virtue of Property (f). Similarly, the (2, 2) block is an $(n-k)$-by-$(n-k)$ matrix which reduces to NN^T by applying Property (f), which further simplifies to I_{n-k} by applying Property (c). This reduces the matrix BB^+ to an identity matrix of size $(n-k+1)$. Hence, the first MP condition is satisfied.

(ii)
$$B^+B = \left[\begin{array}{c|c} \frac{M^{1/2}C^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}} & M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}} \end{array}\right] \begin{bmatrix} \dot{q}^TM^{1/2} \\ NM^{1/2} \end{bmatrix}$$
$$= \left[\frac{M^{1/2}C^TC\dot{q}\dot{q}^TM^{1/2}}{\dot{q}^TC^TCM\dot{q}} + M^{-1/2}N^TNM^{1/2} - \frac{M^{1/2}C^TC\dot{q}\dot{q}^TN^TNM^{1/2}}{\dot{q}^TC^TCM\dot{q}}\right]$$
$$= \left[\frac{M^{1/2}C^TC\dot{q}\dot{q}^T\left(I - N^TN\right)M^{1/2}}{\dot{q}^TC^TCM\dot{q}} + M^{-1/2}N^TNM^{1/2}\right]$$
$$= \left[\frac{M^{1/2}C^TC\dot{q}\dot{q}^TC^TCM^{1/2}}{\dot{q}^TC^TCM\dot{q}} + N^TN\right] \tag{7.6}$$

To arrive at the last equality of (7.6), Properties (a) and (e) have been used. Clearly, the matrix B^+B is symmetric, and thus, the second MP condition is satisfied.

(iii) $BB^+B = I_{n-k+1}B = B$, which directly follows from Eq. (7.5).

(iv)
$$B^+BB^+ = \left[\frac{M^{1/2}C^TC\dot{q}\dot{q}^TC^TCM^{1/2}}{\dot{q}^TC^TCM\dot{q}} + N^TN\right] \left[\begin{array}{c|c} \frac{M^{1/2}C^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}} & M^{-1/2}N^T - \frac{M^{1/2}C^TC\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}} \end{array}\right]$$

The $B^+\,B\,B^+$ matrix is a 1-by-2 block matrix, where the (1, 1) block is given by

$$\begin{aligned}(1,1) &= \left[\frac{M^{1/2}C^TC\dot{q}\dot{q}^TC^TCMC^TC\dot{q}}{\left(\dot{q}^TC^TCM\dot{q}\right)^2} + \frac{N^T\left(NM^{1/2}C^T\right)C\dot{q}}{\dot{q}^TC^TCM\dot{q}}\right] \\ &= \left[\frac{M^{1/2}C^TC\dot{q}\dot{q}^TC^T\left(CC^T\right)CM\dot{q}}{\left(\dot{q}^TC^TCM\dot{q}\right)^2}\right] \\ &= \left[\frac{M^{1/2}C^TC\dot{q}\left(\dot{q}^TC^TCM\dot{q}\right)}{\left(\dot{q}^TC^TCM\dot{q}\right)^2}\right] \\ &= \left[\frac{M^{1/2}C^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}}\right]\end{aligned} \tag{7.7}$$

In the derivation of (7.7) above, the second term of the first equality drops out by virtue of Property (f), and the first term is simplified by using Properties (d) and (b). Next, the (1, 2) block of the matrix $B^+\,B\,B^+$ is given by

$$\begin{aligned}(1,2) = \Bigg[&\frac{M^{1/2}C^TC\dot{q}\dot{q}^TC^T\left(CM^{1/2}M^{-1/2}N^T\right)}{\dot{q}^TC^TCM\dot{q}} + N^TNM^{-1/2}N^T \\ &- \frac{M^{1/2}C^TC\dot{q}\dot{q}^TC^TC\,M^{1/2}M^{1/2}C^TC\dot{q}\dot{q}^TN^T}{\left(\dot{q}^TC^TC\,M\dot{q}\right)^2} - \frac{N^T\left(NM^{1/2}C^T\right)C\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}}\Bigg]\end{aligned} \tag{7.8}$$

The first and the fourth terms of the (1, 2) block above drop out by virtue of Properties (g) and (f), respectively. When Property (e) is applied to the second term and Property (d) is applied to the third term, the (1, 2) block of $B^+\,B\,B^+$ matrix reduces to

$$(1,2)=\left[M^{-1/2}N^T\left(NN^T\right)\right.$$
$$\left.-\frac{M^{1/2}C^TC\dot{q}\dot{q}^TC^T\left(CC^T\right)CM\dot{q}\dot{q}^TN^T}{\left(\dot{q}^TC^TCM\dot{q}\right)^2}\right]$$
$$=\left[M^{-1/2}N^T-\frac{M^{1/2}C^TC\dot{q}\left(\dot{q}^TC^TCM\dot{q}\right)\dot{q}^TN^T}{\left(\dot{q}^TC^TCM\dot{q}\right)^2}\right]$$
$$=\left[M^{-1/2}N^T-\frac{M^{1/2}C^TC\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}}\right] \tag{7.9}$$

We note that Properties (b) and (c) have been used to simplify the first equality of (7.9). Hence, we obtain $B^+\,B\,B^+$ as

$$B^+BB^+=\left[\frac{M^{1/2}C^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}}\middle|\,M^{-1/2}N^T-\frac{M^{1/2}C^TC\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}}\right]$$
$$=B^+, \tag{7.10}$$

which satisfies the fourth MP condition. Since all four MP conditions are satisfied, we ascertain that the B^+ given by (7.4) is indeed the correct expression for the Moore–Penrose inverse of the matrix B.

Main Result: The control force can now be calculated as

$$F^C(q,\dot{q},t)=M^{1/2}(AM^{-1/2})^+(b-Aa)$$
$$=M^{1/2}B^+\left[b-\begin{bmatrix}\dot{q}^TM\\NM\end{bmatrix}M^{-1}F\right]$$
$$=M^{1/2}B^+\left[\begin{bmatrix}\dot{q}^TF-\beta\left(H-H^*\right)\\NF\end{bmatrix}-\begin{bmatrix}\dot{q}^TF\\NF\end{bmatrix}\right]$$
$$=M^{1/2}B^+\begin{bmatrix}-\beta\left(H-H^*\right)\\ [\mathrm{O}]_{(n-k)\times1}\end{bmatrix}$$
$$=M^{1/2}\left[\frac{M^{1/2}C^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}}\middle|\,M^{-1/2}N^T-\frac{M^{1/2}C^TC\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}}\right]\begin{bmatrix}-\beta\left(H-H^*\right)\\ [\mathrm{O}]_{(n-k)\times1}\end{bmatrix}$$
$$=\left[\frac{MC^TC\dot{q}}{\dot{q}^TC^TCM\dot{q}}\middle|\,N^T-\frac{MC^TC\dot{q}\dot{q}^TN^T}{\dot{q}^TC^TCM\dot{q}}\right]\begin{bmatrix}-\beta\left(H-H^*\right)\\ [\mathrm{O}]_{(n-k)\times1}\end{bmatrix}$$
$$=\frac{-\beta\left(H-H^*\right)}{\dot{q}^TC^TCM\dot{q}}MC^TC\dot{q}$$
$$=\frac{-\beta\left(H(q,\dot{q})-H^*\right)}{\dot{q}_C^TM_C\dot{q}_C}I_CM\dot{q}, \tag{7.11}$$

where $\dot{q}_C^TM_C\dot{q}_C=\sum_{g=1}^{k}\left(m_{i_g}\dot{q}_{i_g}^2\right)$ is twice the kinetic energy of the set of controlled masses. Also, from the third equality of (7.11), we note that whenever an energy stabilization constraint is applied to a mechanical system, the term $\dot{q}^TF$ always drops out as long as the system under consideration is conservative. This concludes our derivation of the explicit nonlinear control force in closed form.

Appendix 3: Origin is a unique isolated equilibrium point

Consider a nonhomogeneous NDOF Toda lattice with fixed–fixed (or fixed–free) boundary conditions. The equilibrium points of the uncontrolled (unconstrained) and the controlled (constrained) system can be calculated by substituting $\dot{q}\equiv\ddot{q}\equiv0$ in Eqs. (2.6) and (5.12), respectively. In both cases, we obtain $F=[\mathrm{O}]_{n\times1}$, where the ith row of this relation can be written as

$$a_{i-1}\left[e^{b_{i-1}(q_i(t)-q_{i-1}(t))}-1\right]=a_i\left[e^{b_i(q_{i+1}(t)-q_i(t))}-1\right],$$
$$i=1,\,2,\,\ldots\,n, \tag{7.12}$$

For the fixed–fixed lattice, (7.12) implies

$$a_i\left[e^{b_i(q_{i+1}(t)-q_i(t))}-1\right]=c(t),\quad i=0,\,1,\,2,\,\ldots\,n, \tag{7.13}$$

so that

$$\frac{1}{b_i}\ln\left[1+\frac{c(t)}{a_i}\right]=q_{i+1}(t)-q_i(t),\ i=0,\,1,\,\ldots\,n. \tag{7.14}$$

Summing over i on both sides of (7.14), we have

$$\sum_{i=0}^{n}\frac{1}{b_i}\ln\left[1+\frac{c(t)}{a_i}\right]=\sum_{i=0}^{n}\left(q_{i+1}(t)-q_i(t)\right)$$
$$=q_{n+1}(t)-q_o(t). \tag{7.15}$$

Since $q_o(t)\equiv q_{n+1}(t)\equiv0$, we have $\sum_{i=0}^{n}\frac{1}{b_i}\ln\left[1+\frac{c(t)}{a_i}\right]=0$, whose only solution is $c(t)=0$. From (7.14) then with $c(t)=0$, we have

$$q_{i+1}(t)-q_i(t)=0,\quad i=0,\,1,\,2,\,\ldots\,n, \tag{7.16}$$

which implies $q_i(t)=0,\,i=1,2,\ldots n$, since $q_o(t)\equiv0$. For the fixed–free case, when $i=n$, $a_n=b_n=0$ and hence (7.12) yields

$$a_i\left[e^{b_i(q_{i+1}(t)-q_i(t))}-1\right]=0,\quad i=0,\,1,\,2,\,\ldots\,n-1, \tag{7.17}$$

as in the fixed–fixed case. And since $q_o(t)\equiv0$, we again obtain $q_i(t)=0,\ i=1,\,2,\,\ldots\,n$. Therefore, in

$2n$-dimensional phase space, the origin O, i.e., $(q, \dot{q}) = (0, 0)$, is a unique and isolated equilibrium point of the uncontrolled (and the controlled) nonhomogeneous NDOF Toda lattice.

Appendix 4: Ω is compact

In this appendix, our aim is to show that the set Ω [described by Eq. (5.14)] is compact. The set Ω can be alternatively conceived as

$$\Omega = H^{-1}([\varepsilon, c]) = \left\{x \in \mathbb{R}^{2n} \mid \varepsilon \le H(x) \le c\right\},$$

where $0 < \varepsilon < H^* < c$, and H^{-1} denotes the pre-image of the energy function H, which is given by

$$\begin{aligned} H(x) &= H(q, \dot{q}) \\ &= \sum_{i=1}^{n+1}\left[\frac{1}{2}m_i\dot{q}_i^2\right] \\ &\quad + \sum_{i=0}^{n}\left[\frac{a_i}{b_i}e^{b_i(q_{i+1}-q_i)} - a_i\left(q_{i+1}-q_i\right) - \frac{a_i}{b_i}\right] \\ &= \sum_i h_i \end{aligned} \tag{7.18}$$

To prove that Ω is compact, we make use of the following result [38,39].

Let $X \subset \mathbb{R}^{2n}$ and $Y \subset \mathbb{R}^{+}$ be Euclidean spaces. A function $H : X \to Y$ is radially unbounded if and only if the pre-image $H^{-1}(K)$ of every compact set $K \subseteq Y$ is compact in X.

To use this result, we need to first establish that the energy H is radially unbounded.

$H : \mathbb{R}^{2n} \to \mathbb{R}^{+}$ is a radially unbounded function

The energy function H is said to be radially unbounded if given any $M \in \mathbb{R}^{+}$, there exists an $R \in \mathbb{R}^{+}$ such that $H(x) > M$ for all $\|x\| > R$ [40]. We use the infinity norm $\|\cdot\|_\infty$ to prove our results. The basic idea behind the approach is to equate each individual term h_i (see Eq. 7.18) of the positive definite energy function H to M and find the supremum among the largest absolute values $|x_i|_{\max}$ of the $2n$-coordinates, such that for each of these terms, $h_i(|x_i|_{\max})$ equals M. This supremum value gives us R, which is the side length of the hypercube in $2n$-dimensional phase space. To find R, we adopt the following algorithm.

Step 1: Find the supremum R_v among the largest absolute values of the n velocities. To this end, we consider the kinetic energy terms $T(\dot{q})$ of the energy function H (7.18). When each kinetic energy term is equated to M, out of all the n terms, the term with the infimum mass $m_{k,\inf} = \inf\{m_1, m_2, \ldots, m_n\}$ gives us the supremum velocity. Therefore, $\frac{1}{2}m_{k,\inf}\dot{q}_k^2 = M$ yields

$$R_v = |\dot{q}_k| + \varepsilon_o = +\sqrt{\frac{2M}{m_{k,\inf}}} + \varepsilon_o, \tag{7.19}$$

where $\varepsilon_o > 0$ is included, so that R_v is strictly greater than the supremum $|\dot{q}_k|$.

Step 2: Find the supremum R_d among the largest absolute values of the n displacements. Consider the potential energy terms $U(q)$ of the energy function H (see Eq. 2.4). When the first term $u_o(q_1)$ is equated to M, for any given $M > 0$, there exist precisely two real values $r_1, r_2 \in \mathbb{R}$ such that $u_o(r_1) = M$ and $u_o(r_2) = M$ (see Fig. 2). This is because $u_o(q_1)$ is positive definite and strictly radially increasing (see Sect. 2). Therefore, given an M, a bound on the maximum value of $|q_1|$ is given by $R_1 = \max\{|r_1|, |r_2|\} + \varepsilon_1$ where $\varepsilon_1 > 0$.

Next, let us equate the second term of the potential energy, $u_1(q_2 - q_1)$, to M. Again, since u_1 is positive definite and strictly radially increasing, for a given M, there exist precisely two real values r_3, r_4 such that $u_1(r_3) = M$ and $u_1(r_4) = M$. Therefore, given an M, a bound on the maximum value of $|q_2|$ is given by $R_2 = R_1 + \max\{|r_3|, |r_4|\} + \varepsilon_2$ where $\varepsilon_2 > 0$. Continuing this recursive process, for the ith term, given an $M > 0$, a bound on the maximum value of $|q_i|$ is given by $R_i = R_{i-1} + \max\{|r_{2i-1}|, |r_{2i}|\} + \varepsilon_i$ where $\varepsilon_i > 0\ \forall\ i$, $R_o = 0$, and $u_{i-1}(r_{2i-1}) = u_{i-1}(r_{2i}) = M$ for $i = 1, 2, \ldots n$.

For a fixed–free lattice, there are only n potential energy terms in the energy expression, and therefore, the supremum among the n displacements is given by $R_d = R_n$. On the other hand, for a fixed–fixed lattice, there are $(n+1)$ terms, and equating the last term of the energy expression $u_n(-q_n)$ to M yields yet another estimate for a bound on the maximum value of $|q_n|$. Once again, since u_n is positive definite and strictly radially increasing, for a given M, there exist precisely two real values r_{2n+1}, r_{2n+2} such that $u_n(r_{2n+1}) = M$ and $u_n(r_{2n+2}) = M$. Therefore, a second estimate for a bound on the maximum value of $|q_n|$ is given by $R_n' = \max\{|r_{2n+1}|, |r_{2n+2}|\} + \varepsilon_{n+1}$, where $\varepsilon_{n+1} > 0$. Thus, for a fixed–fixed lattice, the supremum among the largest absolute values of the n displacements is given by $R_d = \max\{R_n, R'_n\}$.

Step 3: Therefore, given any M, a corresponding bound on the side length R of the hypercube in $2n$-dimensional space is given by $R = \max\{R_v, R_d\} + \Delta$, where $\Delta > 0$. Clearly, for all $\|x\|_\infty > R$, we have $H(x) > M$. Hence, the energy function $H(x)$ is radially unbounded.

Thus, by virtue of the result that we stated at the beginning of this appendix, since $H(x)$ is a radially unbounded function, it follows that the pre-image $H^{-1}(K)$ of every compact set $K = [\varepsilon, c] \in \mathbb{R}^+$ is compact in $\mathbb{R}^{2n}$. But, $H^{-1}([\varepsilon, c])$ is our set Ω, and therefore, set Ω is compact in $\mathbb{R}^{2n}$.

Appendix 5: Actuator positions for which the only invariant set of $\dot{q}_C \equiv 0$ is the origin

In this appendix, we want to show that under certain choices of the actuator placements, the only invariant set belonging to $\dot{q}_C \equiv 0$ is the origin ($q \equiv \dot{q} \equiv 0$). We will show this to be true provided that one or more of the following actuator configurations are adopted.

1. The first mass and/or the last mass in the lattice are among the set of masses that are controlled.
2. A consecutive pair of masses in the lattice is among the set of masses that are controlled, i.e., $i_x, i_y \in S_C$, such that $|i_x - i_y| = 1$.

To ensure that the set defined by $H(q, \dot{q}) = H^*$ is the only globally attracting set in Ω, the invariant set(s) satisfying $\dot{q}_C \equiv 0$ should lie outside Ω. But, by actuating an arbitrary set of k masses out of n masses in the lattice, one cannot always guarantee that this holds true as is shown by the following example.

Consider a three-mass homogeneous Toda lattice with fixed ends. The controlled (constrained) equations of motion of the lattice with a single actuator placed at the second mass of the lattice are given by

$$m\ddot{q}_1 = a\left[e^{b(q_2-q_1)} - 1\right] - a\left[e^{b(q_1)} - 1\right], \tag{7.20}$$

$$m\ddot{q}_2 = a\left[e^{b(q_3-q_2)} - 1\right] - a\left[e^{b(q_2-q_1)} - 1\right] -\lambda_o(H - H^*)m\dot{q}_2, \tag{7.21}$$

$$m\ddot{q}_3 = a\left[e^{b(-q_3)} - 1\right] - a\left[e^{b(q_3-q_2)} - 1\right], \tag{7.22}$$

Since the second mass alone is controlled $S_C = \{2\}$, and when $\dot{q}_C \equiv 0$, we have $\dot{q}_2 \equiv 0$. Consequently, $\ddot{q}_2 \equiv 0$. This reduces Eq. (7.21) to

$$q_3(t) - q_2(t) = q_2(t) - q_1(t) \tag{7.23}$$

Differentiating (7.23) with respect to time t and noting that $\dot{q}_2 \equiv 0$, we obtain $\dot{q}_3 = -\dot{q}_1$. Consequently, $\ddot{q}_3 = -\ddot{q}_1$. Solving this along with Eqs. (7.20) and (7.22) yields $q_2 \equiv 0$, and $q_1 \equiv -q_3$. Hence, there exist sets of invariant orbits described by

$$Q = \{(q_1(t), \dot{q}_1(t), 0, 0, -q_1(t), -\dot{q}_1(t))\}$$

that satisfy $\dot{q}_2 \equiv 0$ and that lie inside the set Ω. And, therefore, besides $H(q, \dot{q}) = H^*$, there are additional invariant sets in E (and hence in Ω) that satisfy $\dot{q}_2 \equiv 0$ and to which the trajectories are confined. Thus, in such a case, one cannot guarantee that the set $H(q, \dot{q}) = H^*$ is globally attracting in Ω. To ensure that the sets of invariant orbits satisfying $\dot{q}_C \equiv 0$ lie outside Ω, the actuators must be placed appropriately, so that $\dot{q}_C \equiv 0$ only yields the set $q \equiv \dot{q} \equiv 0$ (origin O), which lies outside Ω.

Actuator Positions Consider that the ith mass of the Toda lattice is actuated. The constrained equation of motion of the ith mass of the lattice can be explicitly written down as

$$m_i\ddot{q}_i = a_i\left[e^{b_i(q_{i+1}-q_i)} - 1\right] - a_{i-1}\left[e^{b_{i-1}(q_i-q_{i-1})} - 1\right] -\lambda_o(H - H^*)m_i\dot{q}_i \tag{7.24}$$

Since the ith mass is controlled, $i \in S_C$, and when $\dot{q}_C \equiv 0$, we have $\dot{q}_i \equiv 0$. Consequently, $\ddot{q}_i \equiv 0$. This reduces Eq. (7.24) to

$$a_i\left[e^{b_i(q_{i+1}-q_i)} - 1\right] = a_{i-1}\left[e^{b_{i-1}(q_i-q_{i-1})} - 1\right]. \tag{7.25}$$

Differentiating Eq. (7.25) with respect to time t, we obtain

$$a_i b_i e^{b_i(q_{i+1}-q_i)}(\dot{q}_{i+1} - \dot{q}_i) = a_{i-1}b_{i-1}e^{b_{i-1}(q_i-q_{i-1})}(\dot{q}_i - \dot{q}_{i-1}), \tag{7.26}$$

which simplifies to

$$a_i b_i e^{b_i(q_{i+1}-q_i)}(\dot{q}_{i+1}) = a_{i-1}b_{i-1}e^{b_{i-1}(q_i-q_{i-1})}(-\dot{q}_{i-1}). \tag{7.27}$$

Now, if we additionally have either $\dot{q}_{i-1} \equiv 0$ or $\dot{q}_{i+1} \equiv 0$, then we obtain certain simplifying results. In the present case, we derive our results assuming $\dot{q}_{i-1} \equiv 0$, but a similar derivation follows if $\dot{q}_{i+1} \equiv 0$ instead. In Eq. (7.27), if $\dot{q}_{i-1} \equiv 0$, since $a_i b_i e^{b_i(q_{i+1}-q_i)} > 0$, it follows that $\dot{q}_{i+1} \equiv 0$ (and hence $\ddot{q}_{i+1} \equiv 0$). Substituting this into the constrained equation of motion of the $(i+1)$th mass of the lattice gives us

$$a_{i+1}\left[e^{b_{i+1}(q_{i+2}-q_{i+1})}-1\right]=a_i\left[e^{b_i(q_{i+1}-q_i)}-1\right] \tag{7.28}$$

Differentiating Eq. (7.28) with respect to time t gives us

$$\begin{aligned} &a_{i+1}b_{i+1}e^{b_{i+1}(q_{i+2}-q_{i+1})}(\dot{q}_{i+2}-\dot{q}_{i+1}) \\ &\quad = a_i b_i e^{b_i(q_{i+1}-q_i)}(\dot{q}_{i+1}-\dot{q}_i), \end{aligned} \tag{7.29}$$

Again, since $\dot{q}_i \equiv \dot{q}_{i+1} \equiv 0$, as before Eq. (7.29) yields $\dot{q}_{i+2} \equiv 0$ (and consequently $\ddot{q}_{i+2} \equiv 0$). Continuing this process of recursive substitution into the constrained equations of motion of the $(i+2)$th mass, and then the $(i+3)$th mass, and so on and so forth until the nth mass of the lattice, we obtain

$$\dot{q}_k \equiv \ddot{q}_k \equiv 0, \qquad k=i-1,\ i,\ i+1,\ i+2,\ \ldots\ n. \tag{7.30}$$

On the other hand, since $\dot{q}_{i-1} \equiv \ddot{q}_{i-1} \equiv 0$, following the steps in Eqs. (7.24–7.27) for the constrained equation of motion of the $(i-1)$th mass of the lattice, we obtain $\dot{q}_{i-2} \equiv 0$ (and hence $\ddot{q}_{i-2} \equiv 0$). Continuing this process of recursive substitution into the constrained equations of motion of the $(i-2)$th mass, and then the $(i-3)$th mass, and so on and so forth until the first mass of the lattice, we obtain

$$\dot{q}_k \equiv \ddot{q}_k \equiv 0, \qquad k=i,\ i-1,\ i-2,\ i-3,\ \ldots\ 1. \tag{7.31}$$

Thus, for a nonhomogeneous NDOF Toda lattice

1. If $i_x, i_y \in S_C$, and $|i_x - i_y| = 1$, then whenever $\dot{q}_C \equiv 0$, we have $\dot{q}_{i_x} \equiv \dot{q}_{i_y} \equiv 0$ with $|i_x - i_y| = 1$, and for $i = i_x$ in Eqs. (7.24–7.31), we obtain $\dot{q} \equiv \ddot{q} \equiv 0$.
2. For a fixed–fixed lattice, if we actuate the first mass of the lattice, when $\dot{q}_C \equiv 0$, we have $\dot{q}_1 \equiv 0$, then $i = 1$ in Eqs. (7.24–7.30) along with $\dot{q}_o \equiv 0$ yields $\dot{q} \equiv \ddot{q} \equiv 0$. On the other hand, if we actuate the last mass of the lattice, when $\dot{q}_C \equiv 0$, we have $\dot{q}_n \equiv 0$, then $i = n$ in Eqs. (7.24–7.27, 7.31) along with $\dot{q}_{n+1} \equiv 0$ yields $\dot{q} \equiv \ddot{q} \equiv 0$.
3. For a fixed–free lattice, actuating the first mass follows a derivation similar to the fixed–fixed case. On the other hand, if we actuate the last mass of the lattice, when $\dot{q}_C \equiv 0$, we have $\dot{q}_n \equiv 0$, then $i = n$ in Eqs. (7.24–7.27, 7.31) along with $a_n = b_n = 0$ yields $\dot{q} \equiv \ddot{q} \equiv 0$.

From Appendix II, we know that for a nonhomogeneous NDOF Toda lattice, if $\dot{q} \equiv \ddot{q} \equiv 0$, then $q \equiv 0$. Thus, for the three cases discussed above, it follows that the origin O ($q \equiv \dot{q} \equiv 0$) is the only invariant point satisfying $\dot{q}_C \equiv 0$. The set Ω has been constructed in such a way that an open region around the origin O of $\mathbb{R}^{2n}$ is excluded from Ω. Since the origin O lies outside the set Ω, the largest invariant set in E (and hence Ω) is the set defined by $H(q, \dot{q}) = H^*$.

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